Aleksandrov's estimate and its generalizations (. . . and lack thereof)

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The following is based partly on past works with Russell Schwab and with Jun Kitagawa

(both currently at Michigan State, coincidentally)

Two results

1.A nonlocal Aleksandrov-Bakelman-Pucci estimate

Consider the elliptic integro-differential equation

$$Lu(x) := \int_{\mathbb{R}^d} (u(x+h) - u(x))(A(x)\hat{h}, \hat{h})|h|^{-d-\alpha}\, dh$$

where $\alpha \in (0,2)$, $A(x) \geq 0$ and $\mathrm{tr}(A(x)) \geq \lambda$ for all x, and

$$\hat{h} := \frac{h}{|h|}$$

Two results

1.A nonlocal Aleksandrov-Bakelman-Pucci estimate

Given a function $\phi : \mathbb{R}^d \to \mathbb{R}$ and $\alpha \in (0,2)$, define

$$\mathcal{H}_\alpha \phi(x) = C(d,\alpha) \int_{\mathbb{R}^d} (\phi(x+h) - \phi(x)) \hat{h} \otimes \hat{h} |h|^{-d-\alpha} \, dh$$

This we will refer to as the *Fractional Hessian of order* α.

$$\left(\text{Observe that } \lim_{\alpha \to 2} \mathcal{H}_\alpha \phi(x) = D^2\phi(x) + \frac{\Delta \phi(x)}{d+2} I \right)$$

Two results

1.A nonlocal Aleksandrov-Bakelman-Pucci estimate

The operator $Lu(x)$ can be written as

$$\text{tr}(A(x)\mathcal{H}_\alpha u(x))$$

which one should think about as a (poor) non-local imitation of

$$\text{tr}(A(x)D^2u(x))$$

Two results

1.A nonlocal Aleksandrov-Bakelman-Pucci estimate

A function ϕ will be said to be α-convex in D, if

$$\mathcal{H}_\alpha \phi(x) \geq 0 \; \forall \; x \in D,$$

It is not difficult to see the maximum of two α-convex functions is still convex.

Two results

1.A nonlocal Aleksandrov-Bakelman-Pucci estimate

For $u : \mathbb{R}^d \to \mathbb{R}$ non-negative outside D we consider,

$$\phi_u(x) = \max\{\phi(x) \mid \phi \leq u \text{ and } \phi \text{ is } \alpha\text{-convex in } D\}$$

this we will call the α-convex envelope of u.

Two results

1.A nonlocal Aleksandrov-Bakelman-Pucci estimate

Theorem (G.-Schwab, 2012)

Suppose that $Lu(x) \leq f(x)$ in D, $u \geq 0$ in $\mathbb{R}^d \setminus D$. *Then*

$$\|u_-\|_{L^\infty(D)} \leq C(D,\lambda,\alpha)\|f\|_{L^\infty(\Gamma_{\alpha,u})}^{\frac{2-\alpha}{2}}\|f\|_{L^d(\Gamma_{\alpha,u})}^{\frac{\alpha}{2}}$$

where $\Gamma_{\alpha,u} = \{x \in D \mid u = \phi_u\}$, *the contact set of* u *and* ϕ_u.

Two results

2.An Aleksandrov estimate for c-convex functions

For a Riemannian manifold M, define the *quadratic cost*

$$c(x,y) = \frac{1}{2} d(x,y)^2$$

A function $\phi : M \to \mathbb{R}$ is said to be c-affine if

$$\phi(x) = -c(x, y_0) + \alpha_0$$

for some $y_0 \in M$ and some $\alpha_0 \in \mathbb{R}$.

Two results

2.An Aleksandrov estimate for c-convex functions

A function which is the supremum of a family of c-affine functions will be called c-convex

$$\phi(x) = \sup_i \{-c(x, y_i) + \alpha_i\}$$

If u is c-convex, its subdifferential at x is defined as

$$\partial^c u(x_0) := \{y \mid \exists\ \alpha \text{ s.t. } \begin{array}{l} u(x) \geq -c(x,y) + \alpha\ \forall\ x \\ u(x_0) = -c(x_0, y) + \alpha \end{array} \}$$

Two results

2.An Aleksandrov estimate for c-convex functions

Problem: Determine the class of costs for which an estimate of the following type holds

$$\|(u-\phi)_-\|_{L^\infty(D)} \leq C|D|^{\frac{1}{d}}|\partial^c u(D)|^{\frac{1}{d}}$$

Two results

2.An Aleksandrov estimate for c-convex functions

Theorem (Figalli-Kim-McCann 2013, G.-Kitagawa 2014)

Essentially, the costs for which the above estimate holds are those satisfying the A3w condition of Ma-Trudinger-Wang.

Two results

These seemingly unrelated results are different generalizations of one of the most consequential (and in my opinion, most underrated) facts in convex geometry, the Aleksandrov estimate.

This talk

1. What is the Aleksandrov estimate?
2. Two illustrations of the Aleksandrov estimate
   - Uniformly elliptic operators with rough coefficients
   - $C^{1,\alpha}$ theory for the real Monge-Ampère equation
3. Theories that could use an Aleksandrov-type estimate
4. New Aleksandrov-type estimates
   - In optimal transport and geometric optics
   - In integro-differential equations
5. A non-local Jacobian equation (if there is time)

1. What is the Aleksandrov estimate?

The Aleksandrov estimate

Theorem (Aleksandrov)

Given a convex body D and $h : D \to \mathbb{R}$ convex with $h\mid_{\partial D} = 0$,

$$\|h\|_{L^\infty(D)} \leq C_d |D|^{\frac{1}{d}} |\nabla h(D)|^{\frac{1}{d}}$$

The Aleksandrov estimate

The (reverse) Blashcke-Santaló inequality

For a convex set $D \subset \mathbb{R}^d$ its **polar dual** is the set

$$D^* = \{y \in \mathbb{R}^d \mid x \cdot y \leq 1 \forall\, x \in D\}$$

The Aleksandrov estimate

The (reverse) Blashcke-Santaló inequality

Theorem (Blashcke-Santaló)

If the center of mass of D is at the origin, then

$$|D||D^*| \geq c_d$$

for some dimensional constant c_d.

The Aleksandrov estimate

The (reverse) Blashcke-Santaló inequality

If h is convex, $h = 0$ on ∂D, and $h = -1$ at x_c = center of D,

$$\nabla h(D) = D^*$$

The Aleksandrov estimate

If $h : D \to \mathbb{R}$ is a convex function and $h = 0$ on ∂D, then

$$\|h\|_{L^\infty(D)} \leq C_d |D|^{\frac{1}{d}} |\nabla h(D)|^{\frac{1}{d}}$$

The Aleksandrov estimate

If h is twice differentiable, then

$$|\nabla h(D)| = \int_{\nabla h(D)} dx = \int_D \det(D^2 h)\, dx$$

Therefore,

$$\|h\|_{L^\infty(D)} \leq C_d |D|^{\frac{1}{d}} \left(\int_D \det(D^2 h)\, dx \right)^{\frac{1}{d}}$$

The Aleksandrov estimate

This estimate is essential for the folowing theorems...

- The Krylov-Safonov estimates for non-divergence equations

$$\mathrm{tr}(A(x)D^2u) = f \Rightarrow \|u\|_{C^\alpha} \leq C(\|u\|_{L^\infty}, A, f)$$

- The Evans-Krylov theorem for convex elliptic equations

$$F(D^2u) = f \Rightarrow \|u\|_{C^{2,\alpha}} \leq C(\|u\|_{L^\infty}, F, f)$$

- Caffarelli's estimates for optimal transport maps in $\mathbb{R}^n$

$$\det(D^2u(x)) = \frac{f(x)}{g(\nabla u(x))} \Rightarrow x + \nabla u(x) \text{ is } C^\alpha \text{ and injective}$$

The Aleksandrov estimate

... also essential for these theorems

- Stroock and Varadhan's solution of the Martingale problem
- Nirenberg-Varadhan Strong maximum principles
- Caffarelli's $W^{2,p}$ estimates for fully nonlinear equations
- Stochastic homogenization
  (Caffarelli-Souganidis-Wang, Armstrong-Smart, Schwab)

2. Two illustrations of the Aleksandrov estimate

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients
2. $C^{1,\alpha}$ theory for the real Monge-Ampère equation

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

Consider a domain $D \subset \mathbb{R}^d$ and an operator

$$Lu(x) = \mathrm{tr}(A(x)D^2u(x))$$

where $A(x)$ is a *diffusion matrix* in D, meaning $A(x)$ is symmetric and there is $\lambda > 0$ such that

$$A(x) \geq \lambda I \;\forall\; x$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

$$Lu(x) = \text{tr}(A(x)D^2u(x))$$

If $A(x)$ is Hölder continuous, then for any $p > d/2$ we have

$$\begin{cases} Lu & = f \text{ in } D \\ u & = 0 \text{ on } \partial D \end{cases} \Rightarrow \|u\|_{L^\infty(D)} \leq C(A,D,p)\|f\|_{L^p(D)}$$

Here, in fact, we have $C(A,D,p) = C(\lambda, \|A\|_{C^\alpha}, \alpha, D, p)$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

What about nonlinear elliptic equations?

Consider a family of matrices $\{A_i\}_i$ such that $A_i \geq \lambda I$.
The Bellman operator associated to this family is defined by

$$F(D^2u) = \inf_i \{\mathrm{tr}(A_i D^2 u)\}$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

If u solves $F(D^2u) = f(x)$, then one may consider

$$A(x) := A_{i(x)}, \;\; i(x) := \text{argmin}\{i \to \text{tr}(A_i D^2 u(x))\}$$

One expects u to also solve the linear equation

$$\text{tr}(A(x)D^2u(x)) = f(x)$$

A priori all we know about $A(x)$ is that $A(x) \geq \lambda I$ for all x.

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

An important fact about PDE has been recognized since at least the time of De Giorgi's and Nash's independent solutions of Hilbert's 19th problem.

Namely:

Any scalar solving a nonlinear PDE solves a linear PDE with potentially discontinuous coefficients

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

$$\operatorname{div}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = 0$$

$$\det(D^2 u) = 1$$

$$\inf_{A \in \mathcal{C}} \operatorname{tr}(A D^2 u) = f$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

Question

If we are given $A(x)$, and u solving

$$\text{tr}(A(x)D^2u(x)) = f \text{ in } D, \; u = 0 \text{ on } \partial D$$

Is there an estimate of the form

$$\|u\|_{L^\infty(D)} \leq C\|f\|_{L^p(D)}$$

where the constant C does not depend on how smooth $A(x)$ is?.

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

Suppose D is convex and $u : D \to \mathbb{R}$ is such that

$$\operatorname{tr}(A(x)D^2u) \leq f \text{ in } D, \ u \geq 0 \text{ on } \partial D.$$

Then, the **Aleksandrov-Bakelman-Pucci estimate** says

$$\|u_-\|_\infty \leq C_d \lambda^{-1} |D|^{\frac{1}{d}} \|f\|_{L^d(\Gamma_u)}$$

Here, $\Gamma_u := \{x \mid u(x) = h_u(x)\}$, $h_u :=$ convex envelope of u.

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

This estimate relies on the Aleksandrov estimate!

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

A key geometric fact about h is that

$$\det(D^2 h) = 0 \text{ in } \{u > h\}$$

Therefore, by the Aleksandrov estimate

$$\|u_-\|_{L^\infty}^d \leq C_d |D| \int_{\Gamma_u} \det(D^2 h)\, dx$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

We now use an important property of the determinant

$$\det(M)^{\frac{1}{d}} = \min\left\{\tfrac{1}{d}\mathrm{tr}(BM) \;\mid\; \text{ where } B > 0 \text{ and } \det(B) = 1\right\}$$

Applying this to $M = D^2h$ and $B = \det(A)^{-\frac{1}{d}}A$,

$$\det(D^2h) \leq \tfrac{1}{d^d}\det(A)^{-1}\left(\mathrm{tr}(AD^2h)\right)^d$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

In the set $\Gamma_u = \{u = h\}$ we have

$$0 \leq \mathrm{tr}(AD^2h) \leq \mathrm{tr}(AD^2u) \leq f(x)$$

Then,

$$\det(D^2h) \leq \tfrac{1}{d^d\lambda^d} f(x)^d \text{ in } \Gamma_u$$

$$\int_D \det(D^2h)\, dx = \int_{\Gamma_u} \det(D^2h)\, dx \leq \tfrac{1}{d^d\lambda^d} \int_{\Gamma_u} f(x)^d\, dx$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

Putting everything together, we arrive at

$$\|u_-\|_{L^\infty(D)} \leq \tfrac{1}{d\lambda} C_d |D|^{\frac{1}{d}} \left(\int_{\Gamma_u} f(x)^d \, dx \right)^{\frac{1}{d}}$$

Two illustrations of the Aleksandrov estimate

1. Uniformly elliptic operators with rough coefficients

A key ingredient in the ABP is the **gradient map**

$$x \mapsto \nabla u(x)$$

the image of which controls $\|u\|_\infty$, and whose Jacobian in turn can be estimated in terms of

$$\mathrm{tr}(A(x)D^2u)$$

and this was thanks to the extremal nature of the determinant

$$\det(M)^{1/d} = \inf\{\tfrac{1}{d}\mathrm{tr}(BM) \mid B > 0 \text{ and } \det(B) = 1\}$$

Two illustrations of the Aleksandrov estimate

2.$C^{1,\alpha}$ theory for the real Monge-Ampère equation

Consider D and $u : D \to \mathbb{R}$ convex, such that

$$\lambda \leq \det(D^2 u) \leq \Lambda$$

for two constants $\lambda, \Lambda > 0$.

This equation we will understand in a weak sense, namely

$$\lambda |E| \leq |\partial u(E)| \leq \Lambda |E|$$

for any Borel set $E \subset D$.

Two illustrations of the Aleksandrov estimate

Theorem (Caffarelli)

If u is as above, and $\partial u(D)$ is convex, then u is strictly convex and $C^{1,\alpha}$ in the interior of D.

The proof relies on studying the shape of the convex sets

$$S_r(x_0) = \{u(x) \leq \ell_{x_0}(x) + r\}$$

where $\ell_{x_0}(x) = u(x_0) + p \cdot (x - x_0)$ and $p \in \partial u(x_0)$.

Two illustrations of the Aleksandrov estimate

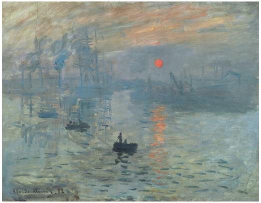
2.$C^{1,\alpha}$ theory for the real Monge-Ampère equation

Theorem (Caffarelli)

The eccentricity of the convex sets $S_h(x_0)$ is controlled as $h \to 0$

Two illustrations of the Aleksandrov estimate

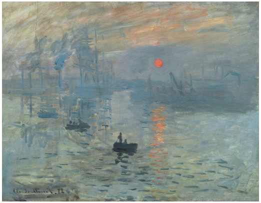
2. $C^{1,\alpha}$ theory for the real Monge-Ampère equation



(A, shall we say, *impressionistic* overview of the proof)

Two illustrations of the Aleksandrov estimate

2.$C^{1,\alpha}$ theory for the real Monge-Ampère equation



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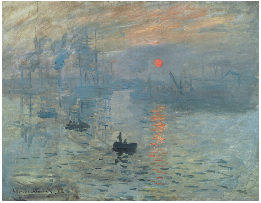
Two illustrations of the Aleksandrov estimate

2.$C^{1,\alpha}$ theory for the real Monge-Ampère equation



(A, shall we say, *impressionistic* overview of the proof)

The Aleksandrov estimate says

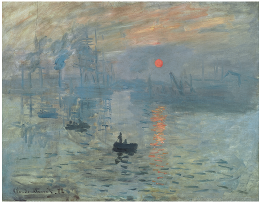
$$r \leq C_d |S_r(x_0)|^{\frac{1}{d}} |\nabla u(S_r(x_0)|^{\frac{1}{d}}$$

Using that $\det(D^2 u) \leq \Lambda$ (in the weak sense)

$$r \leq C_d \Lambda^{\frac{1}{d}} |S_r(x_0)|^{\frac{2}{d}}$$

Two illustrations of the Aleksandrov estimate

2.$C^{1,\alpha}$ theory for the real Monge-Ampère equation



(A, shall we say, *impressionistic* overview of the proof)

The Aleksandrov estimate says

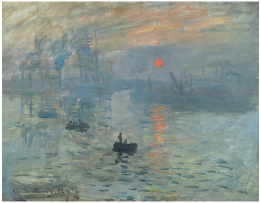
$$r \leq C_d \ell(x_0, S_r) |S_r(x_0)|^{\frac{1}{d}} |\nabla u(S_r(x_0)|^{\frac{1}{d}}$$

Using that $\det(D^2 u) \leq \Lambda$ (in the weak sense)

$$r \leq C_d \ell(x_0, S_r) \Lambda^{\frac{1}{d}} |S_r(x_0)|^{\frac{2}{d}}$$

Two illustrations of the Aleksandrov estimate

2.$C^{1,\alpha}$ theory for the real Monge-Ampère equation



(A, shall we say, *impressionistic* overview of the proof)

Using $\det(D^2u) \geq \lambda$ one shows $r \geq C_d \lambda^{\frac{1}{d}} |S_r(x_0)|^{\frac{2}{d}}$. These estimates combine to give a lower bound on

$$\ell(x_0, S_r)$$

from where the "eccentricity" of $S_r(x_0)$ can be controlled.

3. Theories that could use an Aleksandrov-type estimate

Theories that could use an Aleksandrov-type estimate

Hamilton-Jacobi-Bellman-Isaac equations

$$\mathcal{I}(u, x) = 0$$

Complex Monge-Ampère equation

$$\det\left(\frac{\partial^2 u}{\partial z_j \bar{\partial}_k}\right) = f$$

Prescribed σ_k-equation

$$\sigma_k(D^2 u) = f$$

Theories that could use an Aleksandrov-type estimate

Hamilton-Jacobi-Bellman-Isaac equations

$$\mathcal{I}(u, x) = 0$$

Theories that could use an Aleksandrov-type estimate

Complex Monge-Ampère equation

$$\det \left(\frac{\partial^2 u}{\partial z_j \bar{\partial}_k} \right) = f$$

Theories that could use an Aleksandrov-type estimate

An open problem
Prove an estimate of the type

$$\|\nabla u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_\infty, \lambda, \Lambda)$$

for a plurisubharmonic $u : B_1(\subset \mathbb{C}^n) \to \mathbb{R}$ solving

$$\det\left(\frac{\partial^2 u}{\partial z_j \bar{\partial}_k}\right) = f,\ \lambda \leq f \leq \Lambda$$

4. New Aleksandrov-type estimates

New Aleksandrov-type estimates

In optimal transport

The Monge-Kantorovich optimal transport problem consists on minimizing

$$\inf\left\{\int_M c(x,T(x))\,d\mu(x) \;\mid T_{\#}\mu=\nu\right\}$$

where μ,ν are given probability measures in M.

An important result of Brenier ($M=\mathbb{R}^d$) and Gangbo-McCann (general M) roughly says that if $\nu << d\mathrm{Vol}_M$ then the above problem has a minimizer T, given by

$$T(x)=\exp_x^c(\nabla u(x)),\; u:M\to\mathbb{R}\;\; c\text{-convex}$$

New Aleksandrov-type estimates

In optimal transport

The potential function u solves a Monge-Ampère type equation

$$\det(\nabla^2 u + A_c(x, \nabla u(x))) = \psi_c(x, \nabla u(x))$$

A fundamental object is a tensor, discovered by Ma-Trudinger-Wang, which governs the smoothness of solutions to the above equation.

New Aleksandrov-type estimates

In optimal transport

Theorem (with Jun Kitagawa, 2014)

Let c be a cost function satisfying the A3w condition of Ma-Trudinger-Wang.

If $u : D \to \mathbb{R}$, $D \subset M$ is c-convex with $u = \phi$ on ∂D, where $\phi(x) = -c(x, y_0) + \alpha$ for some $y_0 \in M$ and $\alpha \in \mathbb{R}$, then

$$(u(x) - \phi(x))_- \leq C\ell(x, D)|D|^{\frac{1}{d}}|\partial^c u(D)|^{\frac{1}{d}}$$

where ℓ is a function such that $\ell(x, D) \to 0$ as $x \to \partial D$.

New Aleksandrov-type estimates

In optimal transport

Theorem (with Jun Kitagawa, 2014)

Let c be a cost function satisfying the A3w condition of Ma-Trudinger-Wang.

If $u : D \to \mathbb{R}$, $D \subset M$ is c-convex with $u = \phi$ on ∂D, where $\phi(x) = -c(x, y_0) + \alpha$ for some $y_0 \in M$ and $\alpha \in \mathbb{R}$, then

$$(u(x) - \phi(x))_- \leq C\ell(x, D)|D|^{\frac{1}{d}}|\partial^c u(D)|^{\frac{1}{d}}$$

where ℓ is a function such that $\ell(x, D) \to 0$ as $x \to \partial D$.

A similar estimate was obtained by Figalli, Kim, and McCann (2013) using different methods.

Later (2019) with Kitagawa we extended this to G-convex functions (Trudinger's Generated Jacobian Equations).

New Aleksandrov-type estimates

In optimal transport

These pointwise estimates are essential in the regularity theory for OT maps with bounded densities, as done in works by Figalli, Kim, and McCann, and in works by G.-Kitagawa.

New Aleksandrov-type estimates

In non-local equations

Moving on, let us talk about nonlocal elliptic equations.

Essentially, all nonlocal elliptic operators are of the form

$$Lu(x) = \int_{\mathbb{R}^d} u(x+h) - u(x) - \nabla u(x) \cdot h \; d\nu_x(h)$$

(at least the linear ones)

For every x, ν_x is a **Lévy measure**, i.e. $\nu_x \geq 0$ and

$$\sup_x \int_{\mathbb{R}^d} \min\{1, |h|^2\} \; \nu_x(dh) < \infty$$

New Aleksandrov-type estimates

In non-local equations

$u : \mathbb{R}^d \to \mathbb{R}$

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \text{p.v.} \int_{\mathbb{R}^d} (u(x+h) - u(x)) C_{d,\alpha} |h|^{-d-\alpha} \, dh$$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u & = f \text{ in } D \\ u & = 0 \text{ in } \mathbb{R}^d \setminus D \end{cases}$$

Then, for every $p > 2d/\alpha$,

$$\|u\|_{L^\infty(D)} \leq C(D,p) \|f\|_{L^p(D)}$$

New Aleksandrov-type estimates

In non-local equations

Fix a nonlocal elliptic opeator L

$$Lu(x) = \int \delta_x u(h) \, d\nu_x(h)$$

Let u be a solution of

$$\begin{cases} Lu & = f \text{ in } D \\ u & = 0 \text{ in } \mathbb{R}^d \setminus D \end{cases}$$

Problem: Under what circumstances do we know that

$$\|u\|_{L^\infty(D)} \leq C \|f\|_{L^p(D)}$$

New Aleksandrov-type estimates

In non-local equations

Cafarelli-Silvestre (2009): Consider an operator of the form

$$Lu = \int_{\mathbb{R}^d} \delta_x u(h) \frac{a(h,h)}{|h|^{d+\alpha}}\, dh,\ \lambda \leq a(x,h) \leq \Lambda$$

$$a(x,h) = a(x,-h)$$

Then, given $Lu \leq f$ in B_1 and $u \geq 0$ in $\mathbb{R}^d \setminus B_1$

$$\|u_-\|_{L^\infty(D)} \lesssim_{d,\lambda,\Lambda} \left(\sum_j \|f\|_{L^\infty(Q_j)}^d |Q_j| \right)^{\frac{1}{d}}$$

where Q_j is a special family of cubes covering of the set Γ_u.

New Aleksandrov-type estimates

In non-local equations

Lastly, we recall the result with Schwab (2012): Consider an operator of the form

$$Lu = \int_{\mathbb{R}^d} \delta_x u(h) \frac{(A(x)\hat{h}, \hat{h})}{|h|^{d+\alpha}} \, dh, \ \mathrm{tr}(A(x)) \geq \lambda$$

Then,

$$\|u\|_{L^\infty(D)} \leq C \|f\|_{L^\infty(\{u=\Gamma_{\alpha,u}\})}^{1-\frac{\alpha}{2}} \|f\|_{L^d(\{u=\Gamma_{\alpha,u}\})}^{\frac{\alpha}{2}}$$

where $C = C(d, \lambda, \alpha, D)$.

5. A non-local Jacobian equation

A non-local Jacobian equation

To every continuous, bounded $u : \mathbb{R}^d \to \mathbb{R}$ we associate a map,

$$G_u : \mathbb{R}^d \to C^0_*(\mathbb{R}^d) := \{\phi \in C^0(\mathbb{R}^d) \mid \phi(0) = 0\}$$

defined by

$$G_u(x)(y) = u(x+y) - u(x), \;\; \forall\, y \in \mathbb{R}^d$$

This map is a kind of generalized gradient map.

For a generic smooth u, the image $G_u(\mathbb{R}^d)$ is a d-dimensional manifold living inside $C^0_*(\mathbb{R}^d)$

A non-local Jacobian equation

The derivative of G_u at some x corresponds to the vector field

$$h \mapsto \nabla u(x+h) - \nabla u(x),$$

in the sense that given $e \in (T\mathbb{R}^d)_x$ we have

$$(DG_u)_x e = (\nabla u(x+h) - \nabla u(x), e)$$

In particular, a general linear functional ℓ of DG_u has the form

$$\int_{\mathbb{R}^d} \nabla u(x+h) - \nabla u(x) \cdot d\boldsymbol{\nu}(h)$$

where $\boldsymbol{\nu}$ is a vector measure.

A non-local Jacobian equation

Lemma

Given a Lévy operator,

$$L(u, x) = \int_{\mathbb{R}^d} \delta_x u(y)\, \nu(dy)$$

There is a linear functional ℓ over the space of operators $\mathbb{R}^d \mapsto C^0_*(\mathbb{R}^d)$ such that

$$L(u, x) = \ell((DG_u)_x)$$

In such a case, we shall say ℓ is a **positive** functional.

A non-local Jacobian equation

A rough idea of the proof

A non-local Jacobian equation

Given $D \subset \mathbb{R}^d$, the d-dimensional Hausdorff measure of the manifold $G_u(D)$ is equal to

$$\int_D J((DG_u)_x)\, dx$$

here, $J(M)$ denotes the Jacobian

$$J(M) := \left(\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \|Mx\|_X^{-d}\, d\sigma(x) \right)^{-1}$$

defined for any linear map $M : \mathbb{R}^d \to C^0_*(\mathbb{R}^d)$.

A non-local Jacobian equation

Question:
Is there a result akin to Aleksandrov's estimate for convex functions that involves the integral

$$\int_D J(DG_u(x))\, dx \; ?$$

Thank you!

Comments / Questions / Suggestions
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