MATH 697 FALL 2017

Problem Set 1: Due Thursday Week 3

(1) Let $g : [a,b] \mapsto \mathbb{R}$ be a continuous function such that

$$\int_a^b g(x)\phi''(x)\,dx = 0,$$

for any ϕ which is twice differentiable and compactly supported in (a,b).

(a) Assume that g is itself twice differentiable, show that in this case necessarily $g'' \equiv 0$.

(b) Let K be a $C^\infty(\mathbb{R})$ function such that

$$0 \leq K(x)\ \ \forall\, x, \quad K(x) = 0 \text{ if } x \notin (-1,1), \quad \int_{-1}^1 K(x)\,dx = 1.$$

Then, for $\delta > 0$ define $K_\delta(x) = \frac{1}{\delta}K_\delta(x)$ and $g_\delta(x) := \int_a^b g(y)K_\delta(x-y)\,dy$. Show that if $x \in (a+\delta, b-\delta)$, then g_δ is twice differentiable near x and that $g_\delta''(x) = 0$.

(c) Show that as $\delta \to 0^+$, $g_\delta \to g$ uniformly in every closed interval contained in (a,b).

(d) Conclude g must be an affine function, that is, $g(x) = mx + p$ for some m and p.

(2) We are given N points $0 = x_1 < \ldots < x_N = 1$ in the interval $[0,1]$, and N numbers $y_1, \ldots, y_N$. Fix $\lambda > 0$, define for each $f : [0,1] \mapsto \mathbb{R}$ with a continuous second derivative, the functional

$$\mathcal{J}(f) = \sum_{i=1}^N |f(x_i) - y_i|^2 + \lambda \int_0^1 |f''(x)|^2\,dx.$$

Let f_0 be such that $\mathcal{J}(f_0) \leq \mathcal{J}(f)$ for all f. Then,

(a) Show that if ϕ is smooth and compactly supported in (x_i, x_{i+1}), then

$$\int_{x_i}^{x_{i+1}} f_0''(x)\phi''(x)\,dx = 0,$$

and conclude that $f_0''(x)$ must be an affine function in each interval (x_i, x_{i+1}). *Hint:* Observe that $\mathcal{J}(f_0 + s\phi)$, considered as a function of s, has a local minimum at $s = 0$.

(b) From the previous step, conclude that f_0 must be a polynomial of degree at most 3 when restricted to each interval (x_i, x_{i+1}).

(3) From HTF: Exercise 2.5.

(4) From HTF: Exercise 2.7.

(5) Produce code to accomplish the following: considering $f(x) = 2x - 5$, generate 100 data points of the form (x_i, y_i), where the x_i are equally spaced in the interval $[-10, 10]$, and where for each i,

$$y_i = f(x_i) + \varepsilon_i,$$

the ε_i being i.i.d. random variables. Do this for the following three cases: first with $\varepsilon_i = \pm 1$ with equal probability, second with ε_i being a number uniformly distributed in $[-1,1]$, and third ε_i being distributed according to the standard normal distribution.

Make three plots, where in each case you plot the resulting data (x_i, y_i) together with the graph for the line $y = f(x)$.