Math 623
Fall 2015

Problem Set # 9

(1) Let $1 < p < \infty$ and$f \in L^p((0,\infty))$. Define

$$F(x) := \frac{1}{x}\int_0^x f(y)\, dy, \quad x \in (0,\infty)$$

Prove *Hardy's inequality*:

$$\|F\|_{L^p((0,\infty))} \leq \frac{p}{p-1}\|f\|_{L^p((0,\infty))}$$

Hint: Suppose that $f \geq 0$ and that f is continuous with a compact support in $(0,\infty)$. Note that integration by parts implies that

$$\int_0^\infty F^p(x)\, dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\, dx$$

Note that $xF' = f - F$ and apply Hölder's inequality to $\int F^{p-1}f\, dx$.

(2) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} |x|^{-1}(\log(|x|))^{-2} & \text{if } |x| \leq 1/2 \\ 0 & \text{if } |x| > 1/2 \end{cases}$$

Prove 1) $f \in L^1(\mathbb{R})$ 2) $f^* \notin L^1_{\text{loc}}$. *Hint:* Prove that in this case there is some $c > 0$ such that

$$f^*(x) \geq c|x|^{-1}(-\log(|x|))^{-1} \quad \text{when } |x| \leq 1/2.$$

(3) Let $f \in L^1(\mathbb{R})$, define

$$f_+^*(x) := \sup_{h>0}\int_x^{x+h} |f(y)|\, dy$$

Let $E_\alpha := \{x \in \mathbb{R} \mid f_+^*(x) > \alpha\}$. Show that

$$m(E_\alpha) = \frac{1}{\alpha}\int_{E_\alpha} |f(y)|\, dy$$

Hint: Use the sun rising lemma (Lemma 3.5 in Stein-Shakarchi) and apply it to the function $F(x) = \int_0^x |f(y)|\, dy - \alpha x$.

(4) Let $\{K_\delta\}_\delta$ be an approximation to the identity and $f \in L^1(\mathbb{R}^d)$. Show there is a constant independent of f and x such that

$$\sup_{\delta>0} |(K_\delta * f)(x)| \leq cf^*(x)$$

(5) Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2\sin(x^{-2})$ for $x \neq 0$ and $F(0) = 0$. Show that $F'(x)$ exists for every x but $F' \notin L^1([-1,1])$.

(6) Show that the Cantor-Lebesgue function (see Stein-Shakarchi, Chapter 1 exercise #2 and Chapter 3 p. 125) is **not** absolutely continuous. *Hint:* Use the fact that F is constant on each connected componenet of the complement of the Cantor set, a set of measure of zero.

(7) * A family of functions $\mathcal{A} \subset L^1(\mathbb{R}^d)$ is said to be **uniformly integrable** if given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$m(E) < \delta \Rightarrow \left| \int_E |f| \, dx \right| < \varepsilon \;\; \forall \, f \in \mathcal{A}.$$

Prove that a) Every finite family of functions in $L^1(\mathbb{R}^d)$ is uniformly integrable. b) if $\{f_n\}_n$ is a sequence of functions such that $|f_n(x)| < \infty$ a.e. for every n, $f_n(x) \to f(x)$ a.e. to some function f and $\{f_n\}_n$ is uniformly integrable then $f \in L^1(\mathbb{R}^d)$ and $f_n \to f$ in $L^1(\mathbb{R}^d)$.

(8) * Let $F \in L^1(\mathbb{R})$ be such that there is some $c > 0$ such that

$$\int_{\mathbb{R}} |F(x+h) - F(x)| \, x \leq c|h| \;\; \forall \, h \in \mathbb{R}$$

Show that F must be a function of bounded variation.