Math 623
Fall 2015

Problem Set # 8

(1) (The Saga of the Change of Variables Formula, Part 3 and Ending)
  (a) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a differentiable mapping. The Jacobian of T, often denoted $JT(x)$, is a real valued function defined as

$$JT(x) := |\det(DT(x))|$$

  Let E be a measurable set. Show that,

$$m(T(E)) = \int_E JT(x)\, dx$$

  (b) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be a measurable functions and $T : \mathbb{R}^d \to \mathbb{R}^d$ a differentiable mapping (with an inverse, also differentiable). Suppose that $g(T(x)) = f(x)$, then prove that

$$\int_{\mathbb{R}^d} g(x)\, dx = \int_{\mathbb{R}^d} f(x) JT(x)\, dx$$

  Hint: You may use all the results in previous homeworks involving change of variables. For part b), note that part a) gives immediately the case where f is a simple function.

(2) Let $p \in [1, \infty]$. For every $y \in \mathbb{R}^d$ define $\tau_y : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by

$$(\tau_y f)(x) := f(x - y), \quad \forall\, x \in \mathbb{R}^d, \quad f \in L^p(\mathbb{R}^d)$$

Show that $\|\tau_y f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$ for all $f \in L^p(\mathbb{R}^d)$. Then, prove that if $\alpha_1, \ldots, \alpha_n$ are nonngeative, $\alpha_1 + \ldots + \alpha_n = 1$, and $y_1, \ldots, y_n \in \mathbb{R}^d$ then

$$\tilde{f}(x) := \alpha_1 f(x - y_1) + \ldots + \alpha_n f(x - y_n)$$
$$\Rightarrow \|\tilde{f}\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}.$$

(3) Let $f \in L^1(\mathbb{R}^d)$. Show that for any $p \in [1, \infty]$ and any $g \in L^p(\mathbb{R}^d)$ we have

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)},$$

where

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x - y)\, dy.$$

(4) Let $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $f * g$ is a bounded function (compare this with the previous problem).

(5) Let $f \in L^p(\mathbb{R}^d)$, for some $\delta > 0$, consider the function

$$f^{(\delta)}(x) := \frac{1}{m(B_\delta(x))} \int_{B_\delta(x)} f(y)\, dy$$

Prove that $f^{(\delta)}$ is a continuous function for every $\delta > 0$.

(6) Suppose $f \in L^1([0,b])$ and define

$$g(x) = \int_x^b \frac{f(t)}{t}\, dt, x \in (0,b].$$

Prove that $g \in L^1([0,b])$ and

$$\int_0^b g(x)\, dx = \int_0^b f(t)\, dt.$$

(7) Let $a < b$ be real numbers and

$$f(x) := \begin{cases} e^{-\frac{1}{x-a}-\frac{1}{x-b}} & \text{if } x \in (a,b) \\ 0 & \text{if } x \notin (a,b) \end{cases}$$

Show that $f \in C_c^\infty(\mathbb{R})$.

(8) * Let $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $f * g$ is a continuous function in $\mathbb{R}^d$.

(9) * Let f and $f^{(\delta)}$ be as in exercise # 5, with $p = 1$. Prove that

$$\lim_{\delta \to 0^+} \|f - f^{(\delta)}\|_{L^1(\mathbb{R}^d)} = 0$$

Hint: Consider first what happens if $f \in C_c(\mathbb{R}^d)$.

(10) * Let B_1, B_2 be two balls with the same center, and with B_2 strictly contained in B_1. Show there is a function $F \in C^\infty(\mathbb{R}^d)$ such that

$$F \equiv 1 \text{ in } B_2$$
$$F \equiv 0 \text{ outside } B_1$$

Hint: Use a one dimensional function as in exericse 8.