Math 623
Fall 2015

Problem Set # 7

(1) (The Saga of the Change of Variables Formula, Part 2)

(a) A function $T : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be Lipschitz continuous in Ω if there is some number $0 < L < \infty$ such that

$$|T(x) - T(y)| \leq L|x-y|, \;\; \forall \;\; x, y \in \Omega$$

In this case, the number

$$[T]_{\mathrm{Lip}(\Omega)} = \inf_{x,y\in\Omega, x\neq y} \frac{|T(x) - T(y)|}{|x-y|}$$

is called the Lipschitz constant or the Lipschitz seminorm of T in Ω.

(b) Let $E \subset \mathbb{R}^n$ be a set of measure zero, and $T : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz function. Show that $T(E)$ is a set of measure zero.

(c) If $L : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, then the image of every measurable set is measurable.

(2) Let $f : [a, b] \to \mathbb{R}$ be such that f' exists and is continuous in $[a, b]$. Show that f is a Lipschitz function in $[a, b]$. *Hint: Use the mean value theorem.*

(3) Let $K \subset \mathbb{R}^n$ be a compact set. Show that the function $f(x) = d(x, K)$ is Lipschitz with Lipschitz constant 1. *Hint: Do the case $K = \{0\}$ first, use the triangle inequality.*

(4) * Let $E \subset \mathbb{R}_+ := (0, \infty)$ be a Borel set, and define a measure h by

$$h(E) = \int_E \frac{1}{x}\, dx$$

Given $a \in \mathbb{R}$, let $aE := \{ax | x \in E\}$. Show that for any E Borel and any $a \in \mathbb{R}_+$ we have

$$h(E) = h(aE)$$

Hint: Note that $h(aE)$ and $h(E)$ agree when E is an interval.