Math 623
Fall 2015

Problem Set # 6

(1) Let $f \in L^1(\mathbb{R}^d)$ with $f \geq 0$ a.e. Given any Lebesgue measurable set E, define

$$\mu(E) := \int_E f \, dv$$

Show the following absolute continuity property: for any $\varepsilon > 0$ there exists $\delta > 0$ such that if A and B are measurable sets such that

$$m(A \Delta B) \leq \delta$$

then $|\mu(A) - \mu(B)| \leq \varepsilon$ ($m(\cdot)$ denotes -as usual- Lebesgue measure in $\mathbb{R}^d$).

(2) Let $\{x_1, \ldots, x_N\}$ be a finite set of points in $\mathbb{R}^d$ and $\alpha_1, \ldots, \alpha_N$ positive numbers. Define the set function

$$\mu(E) = \sum_{k | x_k \in E} \alpha_k, \quad \text{for any Borel set } E.$$

Show that
(a) This is a σ-additive measure on the σ-Algebra of Borel sets of $\mathbb{R}^d$.
(b) This measure is not absolutely continuous.
(c) There is no function $L^1(\mathbb{R}^d)$ such that $\mu(E) = \int_E f(x) \, dx$ for all E.

(3) Let $f(x, y)$ be a positive, differentiable function defined in the unit disc $D \subset \mathbb{R}^2$ and let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1, \ z = f(x, y)\}$$

Given $A \subset S$, Borel subset of S, define $\pi(A) := \{(x, y) \in D \mid (x, y, f(x, y)) \in A\} \subset D$, and

$$m(A) := \int_{\pi(A)} \sqrt{1 + f_x^2 + f_y^2} \, dxdy$$

Show that $m(\cdot)$ defines a Borel regular measure in S.

(4) Let $f(x, y) = \sqrt{1 - x^2 - y^2}$, defined for $x^2 + y^2 < 1$. Compute $m(S)$ (you are free to use double integrals since this Lebesgue integral agrees with a Riemann integral).

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space. One can define the **completion** of this space as follows. Let $\overline{\mathcal{M}}$ be the collection of all sets of the form $E \cup Z$ with $E \in \mathcal{M}$ and $Z \subset F$ with $F \in \mathcal{M}$ and $\mu(F) = 0$. Also define $\overline{\mu}(E \cup Z) = \mu(E)$. Then:
(a) $\overline{\mathcal{M}}$ is the smallest σ-algebra containing $\mathcal{M}$ and all subsets of elements of $\mathcal{M}$ of measure zero.
(b) The function $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$, and this measure is complete.

(6) Let $(X, \mathcal{M}, \mu)$ be a measure space. Show a) given f, g two measurable functions in X then $f + g$ and fg are also measurable functions b) Given a sequence $\{f_n\}_n$ of measurable functions then $\sup_n f_n(x)$ and $\limsup_n f_n(x)$ are also measurable functions c) If $f(x) = \lim_n f_n(x)$ where $\{f_n\}_n$ is a sequence of measurable functions in X then f is also measurable.

(7) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Show that given any measurable function f in X there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ with $f_n(x) \leq f_{n+1}(x)$ for every x and such that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x.$$

(For the **next three problems**, assume always that you are dealing with a σ-finite measure space $(X, \mathcal{M}, \mu)$).

(8) State and prove the analogue of Egorov's theorem for $(X, \mathcal{M}, \mu)$.

(9) State and prove the analogue of Fatou's lemma for $(X, \mathcal{M}, \mu)$.

(10) State and prove the analogue of the Dominated Convergence Theorem for $(X, \mathcal{M}, \mu)$.

(11) (The Saga of the Change of Variables Formula, Part 1) Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation.
- (a) Show that there is always some $M > 0$ such that $|Lx - Ly| \leq M|x - y| \; \forall \, x, y \in \mathbb{R}^d$.
- (b) Show that if E is compact, then so is $L(E)$. For $d = 2$, provide an example of a linear transformation L and some compact set E such that $L^{-1}(E)$ is no longer compact.
- (c) Show that if E is a F_σ set, so is $L(E)$. *Note: See the discussion before Corollary 3.5 in SS Chapter I for the definition of F_σ sets.*
- (d) Show that if E is a set of measure zero, then $L(E)$ is also a set of measure zero.
- (e) Show that if E is a measurable set then $L(E)$ and $L^{-1}(E)$ are also measurabe sets.

(12) (The Saga of the Change of Variables Formula, Part $1\frac{1}{2}$) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Show that given any measurable set E, then $m(L(E)) = |\det(L)|m(E)$. *Hint: Prove the result when E is a square, then use the results from the previous exercise to deal with a general E.*

(13) *Given a set $E \subset \mathbb{R}^2$ and $y \in \mathbb{R}$, we define the "slice"

$$E^y := \{x \in \mathbb{R} \mid (x, y) \in E\}$$

Show that if E is a Borel set in $\mathbb{R}^2$, then E^y is a Borel set for every y.

(14) *Let $X = \mathbb{R}^2_+$ be the upper half plane, thought of as a subset of $\mathbb{C}$,

$$\mathbb{R}^2_+ = \{z \in \mathbb{C} \mid z = x + iy, \quad x, y \in \mathbb{R}, y > 0\}$$

Given a matrix $A \in \mathrm{SL}(\mathbb{R}^2)$, i.e.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad ab - dc = 1$$

we define a **fractional linear transformation**, $T : X \to X$, by

$$T(z) = \frac{az + b}{cz + d}$$

Then,
- (a) Check that indeed, $T(z) \in X$ if $z \in X$.
- (b) Compute the Jacobian matrix of T in terms a, b, c and d.
- (c) Given two matrices A_1 and A_2, call their respective linear transformations T_1 and T_2. How are the compositions $T_1 \circ T_2, T_2 \circ T_1$ related to the matrices A_1A_2, A_2A_1?
- (d) Show that if E is a set of measure zero, then $T(E)$ and $T^{-1}(E)$ are also sets of measure zero.