Math 623
Fall 2015

Problem Set # 5

(1) Prove Jensen's inequality for integrals: given a convex function $\phi : \mathbb{R} \to \mathbb{R}$ and E such that $0 < m(E) < \infty$, then for *any* measurable function f we have

$$\phi\left(\frac{1}{m(E)}\int_E f(x)\,dx\right) \leq \frac{1}{m(E)}\int_E \phi(f(x))\,dx.$$

(2) Suppose that $0 < m(E) < \infty$. Show that if $f \in L^p(E)$ $(1 < p < \infty)$, then $f \in L^q(E)$ for $1 \leq q \leq p$. *Hint: (approach #1) use Jensen's inequality, (approach #2) use Hölder's inequality.*

(3) Suppose that $f \in L^p(E)$ $1 < p < \infty$. Show there is some number $c > 0$ such that for all $A \subset E$ we have

$$\int_A |f|\,dx \leq c m(A)^{1/p'}$$

where p' is the so called **Hölder dual** to p, which is defined by the relation $\frac{1}{p'} + \frac{1}{p} = 1$. Can you estimate the constant c from above with $\|f\|_{L^p(E)}$?. *Extra*: Show the converse is not true!.*

(4) Suppose that $f \in L^\infty(E)$. In analogy with the previous problem, show there is a constant $c > 0$ such that for measureable $A \subset E$ we have

$$\int_A |f|\,dx \leq c m(A)$$

Note: It turns out that the converse in this case is true!: if f is such that the above inequality holds for some $c > 0$ independent of the set $A \subset E$, then $f \in L^\infty(E)$. This is a highly nontrivial fact that will be answered thanks to two of the most important tools studied later in the course, the Hardy-Littlewood theorem and the Lebesgue Differentiation Theorem.

(5) For some $m > 0$, consider the function

$$f(x) = \frac{1}{|x|^m}, \quad x \in \mathbb{R}^d.$$

Since this function is continuous away from $x \neq 0$ its (improper) Riemann integral is well defined, and you may use standard calculus tools (i.e. polar coordinates) to compute the value of the integral. *Note: You do not need to write out the explicit formula for ω_{d-1}, the surface area of the $d-1$ dimensional sphere.*

(a) Show that $f \in L^p(B_1(0))$ if and only if $m < d/p$.
(b) Show that $f \in L^p(B_1(0)^c)$ if and only if $m > d/p$.
(c) Conclude that $f \notin L^p(\mathbb{R}^d)$ for every value of p.
(d) Compute an explicit formula for $\|f\|_{L^p(B_1(0))}$ and $\|f\|_{L^p(B_1(0)^c)}$ (in the respective cases when they are finite).
(e) Find a formula for the distribution function of f,

$$\lambda_f(t) := m\left(\{x \in \mathbb{R}^d \mid f(x) \geq t\}\right)$$

(6) Provide an example of a function such that $f \notin L^p(\mathbb{R})$ but for which there is a constant $c > 0$ with

$$m\left(x \mid |f(x)| \geq t\right) \leq \frac{c}{t} \quad \forall\, t > 0.$$

(7) (Compare with problem # 10) Let $\phi(x)$ be a function with compact support such that

$$\int_{\mathbb{R}^d} |\phi(x)|^2 \, dx = 1$$

Fix some sequence of vectors $\{y_n\}$ with $\lim_{n\to\infty} |y_n| = \infty$, and define

$$f_n(x) = \phi(x + y_n)$$

Show that

$$\int_{\mathbb{R}^d} |f_n(x)|^2 \, dx = 1 \text{ for all } n, \quad \text{and } \lim_{n\to\infty} f_n(x) = 0 \text{ for all } x \in \mathbb{R}^d.$$

(8) Fix some E with $0 < m(E)$ and $E \subset B_1(0)$. Consider a sequence of functions defined as $f_n(x) := \chi_E$ if n is odd and $f_n(x) := \chi_{B_1(0)\setminus E}$ if n is even. What is the relevance of this example to Fatou's lemma?.

(9) * Let $f \in L^1(E)$ be positive almost everywhere in E, $m(E) < \infty$. Show that the function

$$\phi(p) := \int_E |f(x)|^p \, dx$$

is a differentiable function of p in $(0, 1)$, and find a formula for its derivative.

(10) * Let $\{f_n\}_n$ be a sequence of functions converging a.e. to a function $f(x)$. Suppose that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n|^2 \, dx = \int_{\mathbb{R}^d} |f|^2 \, dx < \infty$$

Show that $\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n - f|^2 \, dx = 0$. *Hint: Use Fatou's lemma.*

(11) * (The Riemann-Lebesgue Lemma). Let $f \in L^1(\mathbb{R}^d)$. Define the Fourier transform of f by

$$\hat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot y} \, dx$$

Let $d = 1$, then show that

$$\lim_{|y|\to\infty} \int_{\mathbb{R}^1} f(x) e^{-2\pi i x\cdot y} \, dx = 0$$

Hint: (approach #1) note that in problem set #1 you answered this in the special case $f(x) = \chi_{(a,b)}(x)$, *then approximate a general* $f \in L^1$ *with step functions.*
(approach #2) note that in any dimension one may write $\hat{f}(y)$ *as*

$$\frac{1}{2} \int_{\mathbb{R}^d} \left(f(x) - f(x - \tilde{y})\right) e^{-2\pi i x\cdot y} \, dx, \quad \tilde{y} := \frac{1}{2} \frac{y}{|y|^2}$$

then use the L^1 *continuity theorem.*