Math 623
Fall 2015

Problem Set # 4

(1) Let $E \subset [0,1]$ be such that for **any** set A (measurable or not)

$$m_*(A) = m_*(A \cap E) + m_*(A \setminus E)$$

Show that such a set E must be a measurable. What about the converse? If E is measurable, does the identity above hold for any set A?. *Hint: To understand the converse, try to modify E so that $A \cap E$ and $A \setminus E$ can be approximated by sets which are a positive distance from each other.*

(2) Let $\{f_k\}_k$ be a sequence of measurable functions all defined in a set E with $m(E) < \infty$. Suppose that for every $x \in E$ there is some number $0 < M_x < \infty$ such that $|f_k(x)| < M_x$ for every k. Show that for any $\varepsilon > 0$, there exists $F \subset E$ closed and $M > 0$ such that $m(E \setminus F) < \varepsilon$ and $|f_k(x)| \leq M$ for a.e. x in F and every k.

(3) Let f, g be measurable functions such that $\int_{\mathbb{R}^d} f^2\, dx$ and $\int_{\mathbb{R}^d} g^2\, dx$ are $< \infty$. Then consider the function of a single variable $s \in \mathbb{R}$

$$P(s) = \int_{\mathbb{R}^d} (f + sg)^2\, dx$$

Show that $P(s)$ is a polynomial of second order in s which has at most a single (possibly double) real root. Compute the discriminant of $P(s)$, how does it depend on f and g?. Combining these, can you find a relation between the integrals $\int_{\mathbb{R}^d} |f(x)g(x)|\, dx$, $\int_{\mathbb{R}^d} f^2\, dx$ and $\int_{\mathbb{R}^d} g^2\, dx$?.

(4) Let $f : (0,1) \to \mathbb{R}$ be measurable and integrable in $(0,1)$. Find the limit

$$\lim_{k\to\infty} \int_0^1 x^k f(x)\, dx$$

(5) Let $f : E \to \mathbb{R}$ be a measurable function over a set E of finite measure. If $\int_E |f(x)|\, dx < \infty$, show that for any $\varepsilon > 0$ there exists some $M > 0$ such that

$$\int_E |f(x) - f_M(x)|\, dx \leq \varepsilon,$$

where f_M defines the chopped function

$$f_M(x) = f(x)\chi_{\{x:|f(x)|\leq M\}}(x).$$

(6) Consider the function $\phi(x)$ defined by

$$\phi(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise .} \end{cases}$$

Let $\{r_n\}$ be an enumeration of the rational numbers. Define the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n}\phi(x - r_n)$$

Show: 1) f is integrable in $\mathbb{R}$ 2) For a.e. x, the series converges ansd is finite, i.e. $f(x) < \infty$ for a.e. 3) given **any** interval (a, b), one has

$$\sup_{(a,b)} f(x) = \infty$$

(7) Suppose $f(x)$ is a nonngeative, integrable function in $\mathbb{R}^d$. If $\delta = (\delta_1, \delta_2, \dots, \delta_d)$ is a d-tuple of positive numbers, define the rescaled function

$$f^{(\delta)}(x) = f(\delta_1 x_1, \dots, \delta_d x_d)$$

Show that $f^{(\delta)}$ is also integrable, and

$$\int_{\mathbb{R}^d} f^{(\delta)}(x)\, dx = (\delta_1 \dots \delta_d) \int_{\mathbb{R}^d} f(x)\, dx.$$

(8) Given $f(x)$ $(x \in \mathbb{R}^d)$, its **distribution function** is a function $\lambda_f(t) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\lambda_f(t) := m(S_t), \quad S_t := \{x : |f(x)| > t\}.$$

If $f(x)$ is a nonnegative **simple** function. Show that: 1) $\lambda_f(t)$ is a nonincreasing step function (and thus integrable) 2) the following identity holds

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \lambda_f(t)\, dt$$

Hint: If $f = \sum_1^N \alpha_n \chi_{E_n}$, express the step function $\lambda_f(t)$ in terms of the α_n, $m(E_n)$

(9) This problem is a refinement of the previous one. If f_n is an nondecreasing sequence of simple nonnegative functions converging to a function f, show 1) $\lambda_{f_n}(t) \to \lambda_f(t)$ for a.e. t 2) the limit

$$\lim_{n\to\infty} \int_0^\infty \lambda_{f_n}(t)\, dt = \int_0^\infty \lambda_f(t)\, dt.$$

Using 1), 2) and the previous problem, show that for any function $f(x)$ then

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty \lambda_f(t)\, dt.$$

(10) *Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) Find an example of a positive (and continuous!) function f on $\mathbb{R}$ so that f is integrable (i.e. $\int_{\mathbb{R}} f(x)\, dx < \infty$), yet

$$\limsup_{x\to+\infty} f(x) = \infty$$

(b) However, if besides being integrable and continuous f is also **uniformly** continuous in $\mathbb{R}$, show that

$$\limsup_{|x|\to+\infty} f(x) = 0$$

Hint: For a), try first building an example that may be discontinuous, i.e. a step function, then modify it to make it continous.