Math 623
Fall 2015

Problem Set # 3

(1) Let $A \subset [0,1]$ be such that for **any** set E (measurable or not)

$$m_*(A) = m_*(A \cap E) + m_*(A \setminus E)$$

Show that such a set A must be a measurable. What about the converse? If A is measurable, does the identity above hold for any set E?. *Hint: To understand the converse, try to modify A so that $A \cap E$ and $A \setminus E$ can be approximated by sets which are a positive distance from each other.*

(2) Prove that every measurable function is the limit a.e. of a sequence of continuous functions. *Hint: Consider first the case of a step function, use problem 6 from Problem Set #1.*

(3) Suppose $f : \Omega \to \mathbb{R}$ is a real valued function with the property that the set $\{x : f(x) \geq r\}$ is measurable for every rational number r. Show that f is a measurable function.

(4) Let $\{f_k\}_k$ be a sequence of measurable functions all defined in a set E with $m(E) < \infty$. Suppose that for every k there is some number $M_k > 0$ such that $|f_k(x)| < M_k$ for a.e. x in E. Show that for any $\varepsilon > 0$, there exists $F \subset E$ closed and $M > 0$ such that $|f_k(x)| \leq M$ for a.e. x in F and every k.

(5) If E and F are measurable, and $m(E) > 0, m(F) > 0$, prove that their Minkowski sum $E + F$ contains a non-empty open interval. *Hint: Is the problem simpler if one of E or F is open?*

(6) 1. Show that the Minkowski sum of $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$ is again a ball (for any r_i, x_i). What happens in the Brunn-Minkowski inequality in this case?.
2. Let K be a convex set, and let K' be the set obtained by some translating and rescaling of K, i.e. for some $x \in \mathbb{R}^d$ and some $\lambda > 0$ $K' = \{\lambda y + x \mid y \in K\}$. Find a formula for $m(K + K')$ in terms of λ and $m(K)$, what does the result say about the Brunn-Minkwoski inequality?.

(7) *(See first last problem in Problem Set #2) Suppose you are give a measurable set $E \subset [0,1]$ such that for any nonempty open sub-interval I in $[0,1]$, both sets $E \cap I$ and $E^c \cap I$ have positive measure. Then, for the function $f := \chi_E$ show that whenever $g(x) = f(x)$ a.e. in x, then g must be discontinuous at every point in $[0,1]$.

Note This exercise provides an example of a measurable function f on $[0,1]$ such that every function g equivalent to f must be discontinuous at every *point.*

(8) *Let C be the Cantor set. Show that $C + C = [0,2]$. *Note: This shows that two sets might be of measure zero, but their sum might have strictly positive measure. Showing a case of strict inequality in Brunn-Minkowski.*

(9) *Suppose $A, B \subset \mathbb{R}^d$ are convex sets such that $m(A+B)^{1/d} = m^*(A) + m^*(B)$. Show that there exists $\lambda > 0$ and $x \in \mathbb{R}^d$ such that $A = \lambda B + x$.