Math 623
Fall 2015

Problem Set # 2

(1) Suppose that $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

(2) Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n := \{x : d(x, E) < 1/n\}$$

Provide a proof for the following two assertions:
(a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

(b) The conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

(3) Given a collection of sets $F_1, F_2, \ldots, F_n$, construct another collection of sets $F_1^*, F_2^*, \ldots, F_N^*$ wit $N = 2^n - 1$, so that

$$\bigcup_{k=1}^{n} F_k = \bigcup_{j=1}^{N} F_j^*$$

so that the collection $\{F_j^*\}_j$ is made out of pairwise disjoint sets and such that for any k we have $F_k = \bigcup_{F_j^* \subset F_k} F_j^*$ for every k.

(4) **(The Borel-Cantelli Lemma).** Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Show that a) $\limsup E_k$ is a measurable set b) $m(E) = 0$. *Hint: Note that* $\limsup E_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

(5) Let $\{f_n\}$ be a sequence of measurable functions defined in $[0, 1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence of positive real numbers $\{c_n\}_n$ such that

$$\lim_{n\to\infty} \frac{f_n(x)}{c_n} = 0 \quad \text{a.e. } x$$

Hint: Observe that the Borel-Cantelli lemma from the previous problem gives you a method for showing something happens everywhere except a set of measure zero.

(6) Show that there is an infinite, decreasing sequence of sets $E_1, E_2, \ldots$ such that $E_\infty := \bigcup_{i=1}^{\infty} E_i$ where $m_*(E_\infty) < \infty$ and $\lim_i m_*(E_i) > m_*(E_\infty)$.

(7) Let $E \subset \mathbb{R}$ be a measurable set with $0 < m(E) < \infty$. Show that for every $\alpha \in (0, 1)$ there is an open interval I such that

$$m(E \bigcap I) \geq \alpha m(I).$$

(8) An alternative definition of measurability for a set E is: "E is measurable if for any $\varepsilon > 0$ there is a **closed** set $F \subset E$ with $m_*(E - F) < \varepsilon$". Prove that this definition of measurability is equivalent to the one in the text.

(9) Here are some properties of the "Minkowski sum" $A + B$ of two sets A, B (recall that $A + B := \{x \mid x = a + b,\ a \in A, b \in B\}$)
  (a) Show that if **at least one** of A and B is open, then $A + B$ is open.
  (b) Show that if **both** of A and B are closed, then $A + B$ is measurable. *Hint: Show that $A + B$ is a F_σ set.*
  (c) Show that $A + B$ might not be closed even though A and B are both closed.

(10) Show an example of sets A and B with $m(A) = m(B) = 0$, but $m(A + B) > 0$.

(11) Suppose E_i $(i = 1, 2)$ are a pair of nonempty compact subsets of $\mathbb{R}^d$ and that $E_1 \subset E_2$ and $0 < m(E_1) < m(E_2)$. Prove that for any number c with $m(E_1) < c < m(E_2)$ there is some set E such that $E_1 \subset E \subset E_2$ and $m(E) = c$.

(12) Show any open set $E \subset \mathbb{R}^d$ can be written as the union of closed cubes, so that $E = \bigcup Q_i$ with the following properties
  (a) The Q_i are non-overlaping, i.e. their interiors are disjoint.
  (b) There are positive constants $0 < c < C$ so that

$$cm(Q_i)^{1/d} \leq d(Q_i, \Omega^c) \leq Cm(Q_i)^{1/d}$$

Note: Observe that for a cube Q, $m(Q)^{1/d}$ is the same as lenght of any of its edges.

(13) * Show that a σ-algebra with infinitely many sets cannot be countable. *Hint: Show first that if the σ-algebra is infinite then it contains a countable sequence of pairwise disjoint sets. Then recall how one can show $[0, 1]$ is uncountable by using a binary represention. See also problem # 3 above.*

(14) * Suppose that E is measurable with $m(E) < \infty$ and

$$E = E_1 \bigcup E_2, \quad E_1 \bigcap E_2 = \emptyset$$

Suppose that $m(E) = m_*(E_1) + m_*(E_2)$, then show E_1, E_2 are both measurable.
Note: In particular, this would show that if $E \subset Q$, where Q is a finite cube, then E is measurable if and only if $m(Q) = m_*(E) + m_*(Q \setminus E)$.

(15) * Construct a measurable subset $E \subset [0, 1]$ such that for every subinterval I, both $E \cap I$ and $I \setminus E$ have positive measure. *Hint: Take a Cantor-type subset of $[0, 1]$ with positive measure (see previous problem), and on each subinterval of the complement of this set, construct another such set, and so on.*