Math 623
Fall 2015

Problem Set # 10

(1) Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is **exactly** $\mathbb{Q}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function. Show that if E is a set of measure zero, then $f(E)$ is also a set of measure zero.

(3) Suppose that ν, ν_1, ν_2 are signed measures on $(X, \mathcal{M})$ and μ is a positive measure also on $(X, (M))$. Prove

(a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.
(b) If $\nu_1 << \mu$, $\nu_2 << \mu$ then $\nu_1 + \nu_2 << \mu$.
(c) $\nu_1 \perp \nu_2$ implies $|\nu_1| \perp |\nu_2|$.
(d) $\nu << |\nu|$.
(e) If $\nu \perp \mu$ and $\nu << \mu$ then $\nu = 0$.

(4) Consider a measure space $(X, \mathcal{M}, \mu)$ and a sequence $\{f_k\}_k$ of measurable functions which are finite μ-a.e. in some set $E \in \mathcal{M}$. Given another measurable function f, we say that $f_k \to f$ **in measure** if for any $\varepsilon > 0$ we have

$$\lim_{k\to\infty} \mu\left(\{x \in E \mid |f(x) - f_k(x)| > \varepsilon\}\right) = 0$$

Show that if f_k converges to f in measure, then there is a subsequence $\{f_{k_j}\}_j$ such that $f_{k_j} \to f$ μ-a.e. in E as $j \to \infty$.