2. Lecture II
Harmonic functions

(1) A quick Review of Calculus Identities
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2.1. **Caclulus Identities.** A quick word from our sponsor: Calculus.

The Leibniz rule(s),

$$\begin{aligned} \nabla(uv) &= v\nabla u + u\nabla v \\ \mathrm{div}(uX) &= \nabla u \cdot X + u\mathrm{div}(X) \end{aligned} \tag{2.1}$$

The Divergence Theorem

$$\int_{\partial\Omega} X \cdot n \; d\sigma(x) = \int_{\Omega} \mathrm{div}(X) \; dx \tag{2.2}$$

"Integration by Parts" AKA Green's identities

$$\int_{\Omega} u \; \mathrm{div}(X) \; dx = \int_{\partial\Omega} u(X \cdot n) \; d\sigma(x) - \int_{\Omega} \nabla u \cdot X \; dx$$
$$\int_{\partial\Omega} v\partial_n u - u\partial_n v \; d\sigma(x) = \int_{\Omega} v\Delta u - u\Delta v \; dx.$$

2.2. **Harmonic functions, Laplace and Poisson's equation.** The "Laplace operator" is the sum of the pure order second order derivatives of a function u,

$$\Delta u = \mathrm{div}(\nabla u) = \partial_{x_1x_1} u + \ldots + \partial_{x_nx_n} u.$$

A function $u : \Omega \to \mathbb{R}$ such that

$$\Delta u = 0$$

is said to be a *harmonic function*. The above equation is known as *Laplace's equation*. If instead of 0 we have some given function $f(x)$, the resulting equation is called Poisson's equation

$$\Delta u = f$$

2.3. **Examples.** **Example 1.** In $\mathbb{R}^2$,

$$u(x_1, x_2) = x_1^2 - x_2^2$$
$$u(x_1, x_2) = x_1 x_2$$

In general,

$$u(x_1, x_2) = Ax_1^2 + Bx_1x_2 + Cx_2^2, \quad \text{with } A + C = 0.$$

Example 2. Think of a complex variable $z = x + iy$, given any polynomial $P(z)$, we can think of

$$\mathrm{Re}(P(z)) \quad \text{and} \quad \mathrm{Im}(P(z))$$

as functions of the point (x, y). These functions are always harmonic, regardless of what the polynomial P is. In particular, this is so for $P(z) = z^n$ for every $n \in \mathbb{N}$. In fact, Example 1 corresponds to the real and imaginary parts of z^2.

Example 2. Consider the set $\Omega = \mathbb{R}^2 \setminus \{0\}$, then

$$V(x) = \log(|x|)$$

is harmonic in Ω.

Example 3. Consider the set $\Omega = \mathbb{R}^3 \setminus \{0\}$, then

$$V(x) = \frac{1}{|x|}$$

is harmonic in Ω.

2.4. **Boundary value problems.** The **Dirichlet Problem** consists of the following: given a bounded, connected domain Ω and continuous functions $f : \Omega \to \mathbb{R}, g : \partial\Omega \to \mathbb{R}$, **find** a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The **Neumann Problem** is similar, except that instead of saying $u = g$ on the boundary, we prescribe the outer normal derivative of u, $\partial_n u$

$$\begin{cases} \Delta u &= f \quad \text{in } \Omega, \\ \partial_n u &= g \quad \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Note: For the Neumann problem is clear that if u is a solution and c is some constant, then $u + c$ is also a solution.

Our goal is to understand well-posedness in the context of the above problems. Thus we have **Three Big Questions**,

(1) (Existence) Does there exist at least one solution u to (2.3) (or (2.4)).
(2) (Uniqueness) Given f and g, is there at most one solution to (2.3) (or (2.4)).
(3) (Stability) Suppose that we are given two other functions $\tilde{f}$ (in Ω) and $\tilde{g}$ (on Ω). If $|f - \tilde{f}|$ and $|g - \tilde{g}|$ are small, does it follow that the (allegedly unique) respective solutions u and $\tilde{u}$ close to each other?, i.e. is $|u - \tilde{u}|$ also small?

Of the three, Uniqueness will be the easiest, followed by Stability. Existence will take some more work.

2.5. **Uniqueness.** We are goint to show that given f and g, there is at most one solution to (2.3) (and the same goes for (2.4)). First, we recall a calculus fact

Lemma 2.1. *Suppose $h(x)$ is continuous and that $h(x) \geq 0$ for all x in some region Ω, and that*

$$\int_\Omega h(x)\, dx = 0.$$

Then, $h(x) = 0$ for all x.

Proof. Suppose that $h(x_0) > 0$ at some x_0 in the interior of Ω. Since h is continuous, by picking $\delta > 0$ small enough the function h will still be strictly positive in the ball $B_\delta(x_0)$ (which is also contained in Ω, by taking δ even smaller). Then we can break the integral in two parts

$$\int_\Omega h(x)\, dx = \int_{B_r(\delta)(x_0)} h(x)\, dx + \int_{\Omega \setminus B_r(x_0)} h(x)\, dx$$

We do not know anything about what happens with the second integral, except that -since $h \geq 0$-, that it must be nonnegative, therefore

$$\int_\Omega h(x)\, dx \geq \int_{B_r(\delta)(x_0)} h(x)\, dx$$

But the integral fo h over $B_r(\delta)$ is strictly positive on account of $h > 0$ in the ball. This contradicts the fact that the integral over all Ω was zero, and from the contradiction we conclude that $h(x_0) = 0$ for every x_0, as we wanted.

□

Theorem 2.2. *(Uniqueness) Problem* (2.3) *has at most one solution and Problem* (2.4) *has at most one solution up to the addition of a constant, i.e. if u and v both solve* (2.4)*, then $u - v$ is a constant.*

Proof. Suppose u and v are two solutions to either (2.3) or (2.4), let $w = u - v$. Our goal is to show that $w = 0$ (for the Dirichlet problem) or that it is in any case a constant (for the Neumann problem).

The function w solves a "homogeneous" boundary problem, i.e.

$$\Delta w = 0 \text{ in } \Omega$$

and

$$w = 0 \text{ (or } \partial_n w = 0) \text{ on } \quad \partial\Omega.$$

Now, since $\Delta w = 0$, it follows that

$$\int_\Omega w \Delta w\, dx = 0$$

On the other hand, integration by parts says that

$$\begin{aligned} \int_\Omega w \Delta w\, dx &= \int_{\partial\Omega} w \partial_n w\, d\sigma(x) - \int_\Omega |\nabla w|^2\, dx \\ &= -\int_\Omega |\nabla w|^2\, dx. \end{aligned}$$

The last identity being thanks to the fact that $w\partial_n w$ is always zero on $\partial\Omega$ (for both the Dirichlet and Neumann problems). Then, $|\nabla w|^2 \geq 0$ and its integral over Ω is zero, in which case Lemma 2.1 says that $|\nabla w|^2 = 0$ in Ω. So, w must be a constant. In the case of the Dirichlet problem, this constant is zero since $w = 0$ on $\partial\Omega$ on that case. The theorem is proved. □

The above is the so called "energy method" proof of uniqueness. Energy referring to the fact that in applications the integral

$$\int_{\Omega} |\nabla u|^2 \, dx$$

often represents energy. There is another method of proving uniqueness, known as the maximum principle or comparison method. We explore it now, and on the way develop the most important property of harmonic functions -the mean value property.

2.6. **The Mean Value Property.**

Theorem 2.3. *Suppose $\Delta u = 0$ in Ω. Then, for any $x_0 \in \Omega$ and any $r > 0$ such that $|x - x_0| < d(x_0, \partial\Omega)$, we have that 1) the average of u over the sphere of radius r centered at x_0 is equal to $u(x_0)$, that is*

$$u(x_0) = \frac{1}{\text{Area}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u(x) \, d\sigma(x)$$

and 2) the same holds if instead we do the average over the entire ball,

$$u(x_0) = \frac{1}{\text{Vol}(B_r(x_0))} \int_{B_r(x_0)} u(x) \, dx.$$

Corollary 2.4. *(Strong Maximum/Minimum principle) If u is harmonic in Ω -as always an open, bounded, connected domain-, then*

$$\max_{\Omega} u = \max_{\partial\Omega} u$$
$$\min_{\Omega} u = \min_{\partial\Omega} u$$

and, if the maximum or minimum of u is achieved at an interior point of Ω, then u must be a constant.

2.7. **The Mean Value Property II: Subharmonic and Superharmonic functions.** A continuous function $u : \Omega \to \mathbb{R}$ is said to be subharmonic if for every $x_0 \in \Omega$ and every r smaller than the distane from x_0 to $\partial\Omega$ we have the inequality

$$u(x_0) \leq \frac{1}{\text{Vol}(B_r(x_0))} \int_{B_r(x_0)} u(x) \, dx$$

If the integral is reversed, we say u is superharmonic. The mean valuen property shows that any harmonic functions is both subharmonic and superharmonic.

Theorem 2.5. *Suppose $\Delta u \geq 0$ in Ω. Then, for any $x_0 \in \Omega$ and any $r > 0$ such that $|x - x_0| < d(x_0, \partial\Omega)$, we have that 1) the average of u over the sphere of radius r centered at x_0 is equal to $u(x_0)$, that is*

$$u(x_0) \leq \frac{1}{\text{Area}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u(x) \, d\sigma(x)$$

and 2) the same holds if instead we do the average over the entire ball,

$$u(x_0) \leq \frac{1}{\text{Vol}(B_r(x_0))} \int_{B_r(x_0)} u(x) \, dx.$$

Moreover, if $\Delta u \leq 0$, the opposite inequalities hold.

Corollary 2.6. *(Strong Maximum principle for subharmonic functions) If u is subharmonic in Ω -again an open, bounded connected domain- then*

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

and, if the maximum of u is achieved at an interior point of Ω, then u must be a constant. Likewise, if u were superharmonic, the same would hold but this time with the minimum instead of the maximum.

Proof. Because u is continuous, it must achieve its maximum at some point x_0 in the closure of Ω, i.e the set formed by Ω plus its boundary $\partial\Omega$. All we need to show is that this point x_0 cannot be in the interior, so that maximum over all Ω ends up being equal to the maximum over $\partial\Omega$.

Suppose this maximum happens at a point x_0 that is not on the boundary. We are goint to show that then u must be a constant, which would prove the corollary (think of why this is so!). Let y be some other point in the interior of Ω, we are going to show that $u(x) = u(y)$.

Since Ω is connected and both x and y lie in its interior we can find a small enough radius $r > 0$ and a sequence of interior points $x_1, x_2, \ldots, x_N$ with and $y = x_N$ such that: each ball $B_r(x_i)$ $(i = 0, 1, \ldots, N)$ lies in the interior of Ω and such that each ball $B_r(x_i)$ contains x_{i+1}, the center of the next ball $B_r(x_{i+1})$ $(i = 0, 1, \ldots, N-1)$.

Now, let us use the fact that the maximum of u is achieved at x_0, it means in particular that $u(x) - u(x_0) \leq 0$ for all x, so that

$$\int_{B_r(x_0)} u(x) - u(x_0)\, dx \leq 0$$

But, since u is subharmonic, this integral is always ≥ 0, so it follows that

$$\int_{B_r(x_0)} u(x) - u(x_0)\, dx = 0$$

Because the integrand has a sign, our old friend Lemma 2.1 guarantees that $u(x) - u(x_0) = 0$ for all x in $B_r(x_0)$. In other words, the maximum of u in Ω is achieved at every point of $B_r(x_0)$.

Now, since x_1 is contained in $B_r(x_0)$ (by the way we picked the points), it follows that the maximum is achieved at x_1 too. But then repeating the above argument we conclude that u is also constant and equal to $u(x_0)$ in $B_r(x_1)$, so in particular the maximum of u is achieved at the point x_2, and so on. Repeating this we will eventually reach the ball containing y, and conclude that $u(y) = u(x_0)$, as we wanted. The corollary is thus proved. ☐

Corollary 2.7. *Consider a domain Ω and two functions u and v such that*

$$\Delta u \geq 0, \quad \Delta v \leq 0 \text{ in } \Omega,$$

and $u \leq v$ in $\partial\Omega$. Then,

$$u \leq v \text{ in } \Omega.$$

Proof. Let $w = u - v$, then $\Delta w = \Delta u - \Delta v \geq 0$ (thanks to the signs of Δu and Δv). Therefore w is subharmonic in Ω. By the strong maximum principle (Corollary 2.6) it follows that the maximum of w over Ω is the same as the maxmimum over $\partial\Omega$, but

$$\max_{\partial\Omega} w = \max_{\partial\Omega} u - v \leq 0,$$

since $u \le v$ on $\partial\Omega$. Thus the maximum of w in Ω is ≤ 0, which means that $u \le v$ everywhere in Ω, as stated. □

Now, we can do the **comparison method** proof of uniqueness, i.e. of Theorem 2.2: as before, if u and v are both solutions to the same problem, it follows that $w = u - v$ solves

$$\begin{aligned} \Delta w &= 0, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

So, $\max_{\partial\Omega} w = \min_{\partial\Omega} w$, by Maximum principle, it follows that $\max_{\Omega} w = \min_{\Omega} w = 0$. Therefore w is zero and u and v are the same function, proving uniqueness.

But this method says much more, it gives us a way of getting **stability**, at least when we perturb just the boundary conditions. Indeed, suppose u and $\tilde{u}$ solve,

$$\begin{cases} \Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{cases} \qquad \begin{cases} \Delta \tilde{u} &= f \text{ in } \Omega, \\ \tilde{u} &= \tilde{g} \text{ on } \partial\Omega. \end{cases}$$

Then, two applications of the comparison principle lead to the inequality

$$\max_{\Omega} |u - \tilde{u}| \le \max_{\partial\Omega} |g - \tilde{g}|,$$

i.e. we have stability with respect to the boundary conditions. Stability with respect to f will be obtained later once we have the **Green** function in our hands.

Next, we will tackle the question of **existence** for (2.3) in the special case where $f = 0$ and Ω is a ball. To this end, it will be useful to understand the Laplacian in polar coordinates.

2.8. **The Laplacian in Polar Coordinates.** In the plane, the Laplacian of u is defined in terms of the second derivatives with respect to the variables x_1, x_2. If we do a change of variables to polar coordinates, i.e $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, where $r > 0, 0 \le \theta < 2\pi$ we can use the chain rule to rewrite $\partial_{x_i} u, \partial_{x_i x_i} u$ in terms of $\partial_r u, \partial_{rr} u$ and $\partial_{\theta\theta} u$, doing this we can arrive at the formula

$$\Delta u = \partial_{rr} u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\partial_{\theta\theta} u$$

Example. If u is defined in $\mathbb{R}^2 \setminus \{0\}$ via $u(r,\theta) = \log(r)$, then with the above formula it is not hard to see that $\Delta u = 0$ for $r > 0$.

Example. Fix $n \in \mathbb{N}$, and consider the function given in polar coordinates by

$$\begin{aligned} u(r,\theta) &= r^n \cos(n\theta), \\ u(r,\theta) &= r^n \sin(n\theta). \end{aligned}$$

Then with our radial coordinates formula, we can see that each u above is harmonic: for instance,

$$\begin{aligned} \Delta\left(r^n \cos(n\theta)\right) &= n(n-1)r^{n-2}\cos(n\theta) + \frac{2}{r} n r^{n-1}\cos(n\theta) + \frac{1}{r^2} r^n(-n^2\cos(n\theta)) \\ &= n^2 r^{n-2}\cos(n\theta) - n^2 r^{n-2}\cos(n\theta) = 0. \end{aligned}$$

As we will see, these class of functions will be important in analyzing the existence of solutions to harmonic functions on a disc.

2.9. **Separation of Variables and Poisson's Kernel for the Ball.** We are going to show the existence of solutions to

$$\Delta u = 0 \text{ in } D_R(0),$$
$$u = g \text{ on } \partial D_R(0).$$

Note that the last example in the previous section provides with the solution to the above problem on the special cases where $g(x) = g(\theta)$ with either $g(\theta) = R^n \cos(n\theta)$ or $g(\theta) = R^n \sin(n\theta)$, in those cases, the solutions are respectively $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$.

We are going to construct the solution in general by combining these special solutions. First, we invoke an important result about functions $g(\theta)$ that happen to be continuous and periodic with period 2π: each such $g(\theta)$ can be written as a **Fourier series**, namely, a series of functions comprised of $\sin(n\theta)$ and $\cos(n\theta)$. Specifically,

$$g(\theta) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\theta) + \beta_n \sin(n\theta)$$

With the boundary expressed as such a series, by linearity we can find the solution by combining (with the same coefficients) the solutions $r^n \cos(n\theta), r^n \sin(n\theta)$ for each of the terms in the series, in other words,

$$u(r, \theta) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} R^{-n}\alpha_n r^n \cos(n\theta) + R^{-n}\beta_n r^n \sin(n\theta)$$

Each term in the series is a harmonic function, so u itself is harmonic. Taking $r = R$ we get back the series expression for g, which shows that $u = g$ on $\partial D_R(0)$.

2.10. **Harnack Inequality.**

Theorem 2.8. *If $\Delta u = 0$ in $B_{2r}(x_0)$ and $u \geq 0$ everywhere, then*

$$\sup_{B_r(x_0)} u \leq C \inf_{B_r(x_0)} u$$

2.11. **The Newtonian Potential.** A problem in the whole space: given a function f in $\mathbb{R}^3$, find a function $u(x)$ such that

$$\Delta u = f \quad \text{in } \mathbb{R}^3.$$
$$u(x) \to 0 \quad \text{as } |x| \to \infty.$$

Theorem 2.9. *The unique solution to the above problem is given by*

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus \{0\}} \frac{f(y)}{|x - y|}\, dy, \quad \forall\, x \in \mathbb{R}^3. \tag{2.5}$$

2.12. **The Green Function.** For the rest of this section, $V(x)$ will denote the function

$$V(x) = -\frac{1}{4\pi|x|}$$

Given a domain Ω, and $x \in \Omega$, denote by $\phi^{(x)}$ the unique solution of

$$\Delta_Y \phi^{(x)}(y) = 0 \qquad \text{in } \Omega,$$
$$\phi^{(x)}(y) = V(x - y) \text{ on } \partial\Omega.$$

Then, we define the Green function for Ω, as the function $G(x,y)$ of two variables $x, y \in \Omega$ given by

$$G(x,y) = V(x-y) - \phi^{(x)}(y) \tag{2.6}$$

Theorem 2.10. *The unique solution to*

$$\begin{cases} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{cases}$$

is given by

$$u(x) = \int_{\partial\Omega} g(y)(\nabla_y G(x,y) \cdot n)\, d\sigma(y) + \int_{\Omega} f(y) G(x,y)\, dy.$$

Remark 2.11. In fact, $G(x,y)$ itself solves the boundary value problem

$$\begin{aligned} \Delta_y G(x,y) &= \delta(y-x) \quad && \text{in } \Omega, \\ G(x,y) &= 0 && \text{if } y \in \partial\Omega. \end{aligned}$$