M534H
INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
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1. Lecture I

1.1. **Notation.** Let us set up quickly some notation for the semester (more notation to be be added later).

- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ will denote respectively the natural, integer and real numbers.

- $\mathbb{R}^n$, n-dimensional Euclidean space $\{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}(\text{real numbers})\}$

- Points (vectors) in $\mathbb{R}^n$ will usually be denoted by $x, y, \ldots$, with subscripts used for their coordinates. So $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

- U, V will denote open subsets of $\mathbb{R}^n$. ∂U will denote the boundary of U, and $\bar{U} = U \cup \partial U$ will denote the closure of U.

- The partial derivatives of a function $u(x_1, \ldots, x_n)$ will be denoted in any of the following ways: $\frac{\partial}{\partial x_i} u, \partial_{x_i} u$, u_{x_i}. Higher order derivatives will then be $\frac{\partial^2}{\partial x_i \partial x_j} u, \partial_{x_i x_j} u, u_{x_i x_j}$ and so on.

- If $k \in \mathbb{N}$, $D^k u(x)$ will denote the set of all derivatives of order k of u. Since there are n^k such derivatives, $D^k u(x)$ can be thought of as a point in $\mathbb{R}^{n^k}$.

- In particular: $D^1 u(x)$ is n-dimensional vector, $D^2 u(x)$ is a $n \times n$ matrix.

- We will also use the ∇ symbol to denote the gradient of u, so $\nabla u(x)$ is just another symbol for $D^1 u(x)$.

- Finally, $\text{div}(X)$ denotes the divergence of a given vector field X. The Laplacian of a function u is denoted by Δu and its defined as: $\Delta u = \text{div}(\nabla u) = \sum\limits_{i=1}^{n} u_{x_i x_i}$.

1.2. **PDEs.** Formally speaking, a **partial differential equation** (PDE) is a relation between a function $u(x)$ and its (partial) derivatives. Thus, a PDE is given by a function F

$$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \ldots \mathbb{R}^{n^k} \to \mathbb{R}, \quad k \in \mathbb{N}$$

and the equation itself poses the problem of finding a **function** $u(x)$ such that

$$F(x, u, Du(x), D^2 u(x), \ldots, D^k u(x)) = 0 \text{ for every } x.$$

Such a u is called a **solution** to the PDE. The number k is called the **order** of the PDE. Most of the time, n will be $1, 2$ or 3.

The PDE is said to be **linear** if F is linear in all of its arguments except for (possibly) the x variable.

The PDE is said to be **semilinear** if F is linear in all of its arguments except for (possibly) x and u.

The PDE is said to be **quasilinear** if F is at least linear with respect to the highest order derivatives.

If a PDE is not necessarily linear, semilinear or quasilinear then it is said to be **fully nonlinear**.

Thus: linear PDEs are a subclass of semilinear PDEs, which are a subclass of quasilinear PDEs, which are a subclass of fully nonlinear PDEs.

1.3. **Examples.** PDEs are ubiquitous all across mathematics, science, and engineering. The variety and complexity of PDEs is as rich as the phenomena studied and modelled in the natural sciences. Here mention a small sample of the most famous ones:

(1) The transport equation: $\partial_t u(x,t) + b \cdot \nabla u(x,t) = 0$

(2) Laplace's Equation: $\Delta u(x) = 0$

(3) Poisson's equation: $-\Delta u(x) = \rho(x)$

(4) The Wave Equation $\partial_{tt} u(x,t) = \Delta u(x,t)$

(5) The Heat Equation $\partial_t u(x,t) = \Delta u(x,t)$

(6) Schrödinger's equation $i\partial_t u(x,t) = -\Delta u(x,t) + V(x)u(x,t)$

(7) The 2d Monge-Ampére equation: $u_{xx}u_{yy} - u_{xy}^2 = 1$

(8) The Eikonal equation $|\nabla u(x)|^2 = 1$

(9) The Euler equation for incompressible, inviscid flow

$$\partial_t U + U \cdot \nabla U = -\nabla p$$
$$\text{div}(U) = 0$$

(10) The Navier-Stokes equation for incompressible, viscous flow

$$\partial_t U + U \cdot \nabla U = \nu \Delta U - \nabla p$$
$$\text{div}(U) = 0$$

(11) Other examples: Maxwell's equations for electromagnetism, Einstein's Field equations from general relativity, the Korteweg-de Vries equation for shallow water waves, the Black-Scholes equation from economics.

For this course, we will be focused mostly on equations related to examples 1 through 6.

1.4. **Solving a PDE.** What does it mean to solve a PDE? As with algebraic equations, it can mean to simply provide a "formula" for a solution, of for all solutions even, if possible. However, such formulas often do not exist or even if they do are too complicated to be of practical use.

Let us illustrate representation formulas for solutions of some PDEs.

Example. Let us describe all the function $u(x, y)$ solving the equation

$$u_{xx}(x, y) = 0$$

Seeing this as an ODE in x with y as a parameter, we guess $u(x, y)$ must be a first order polynomial in x

$$u(x, y) = ax + b, \text{ for constants } a, b.$$

However, the solution could be more complicated. Because when we perform the two integrations in x there are constants -in x, which may be functions of y-. To see this precisely, integrating the PDE with respect to x (with y fixed) gives that

$$u_x(x, y) = a(y), \quad \text{for some } a,$$

but b might depend on y!, so $a = a(y)$. Integrating with respect to x one more time, we get

$$u(x, y) = a(y)x + b(y), \quad \text{for some } b.$$

This describes all solutions to the PDE!.

Exercise. Find the unique function $u(x, y)$ which solves the PDE

$$u_{xx}(x, y) = 0$$

and which is such that $u(0, y) = \sin(y), u_x(0, y) = \cos(y)$ for every y.

Example. Suppose $f(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solves the PDE

$$f_t + cf_x = 0$$

Note that the expression $f_t + cf_x$ can be interpreted as the inner product between the vector (f_x, f_t) and the vector $(c, 1)$. So the fact that f solves the PDE is nothing but the assertion that $f(x, t)$ is constant along lines parallel to $(c, 1)$. In particular

$$f(x, t) = f(x - ct, t - t) = f(x - ct, 0)$$

So, if we know all the values of the solution at $t = 0$ we can recover the values at any other time t, by simply shifting in the space variable.

Exercise. Suppose that f solves the above PDE and define a new function $\tilde{f}$ by $\tilde{f}(x, t) = f(x, -t)$. What PDE does $\tilde{f}$ solve?.

Example. We will see in the next few lectures that if $u(x)$ solves the 3d Poisson's equation

$$-\Delta u = 4\pi G\rho(x), \quad x \in \mathbb{R}^3$$

Then, $u(x)$ has the following integral representation

$$u(x) = \int_{\mathbb{R}^3} \frac{G\rho(y)}{|x-y|}\, dy$$

($|x|$ denotes the length of the vector x).

Exercise. Consider the function

$$U(x) = \frac{1}{4\pi|x|}, \quad x \in \mathbb{R}^3.$$

Compute $U_{x_1x_1}$ when $x \neq 0$. What can you say about ΔU?. *Hint: Use the formula*

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

together with the chain rule to compute $\partial_{x_i}|x|$.

Example. The Monge-Ampére equation gives a simple example of a PDE for which solutions generally do not have a explicit formula. That is the equation

$$u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2 = 1$$

Exercise. Check that $u(x_1,x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ is a solution of the Monge-Ampére equation. Can you provide another (polynomial) example of a solution?