Math 534H
Homework V
(Due Tuesday, April 14th)

(1) Find the solution to the initial value problems

$$\text{a)}\quad \partial_t u + 5\partial_x u = -2u$$
$$u(x,0) = (1-x^2)_+$$

$$\text{b)}\quad \partial_t u + \partial_x u = \cos(2\pi x)$$
$$u(x,0) = 1+\sin(2\pi x)$$

$$\text{c)}\quad \partial_t u - \partial_x u = g(x,t)$$
$$u(x,0) = 1+\sin(2\pi x)$$

In the last problem, $g(x,t)$ is the function which is 1 if $t-2 \leq x \leq t+2$ and 0 otherwise.

(2) Find the solution to the initial value problem

$$\partial_t u + x\partial_x u = 0$$
$$u(x,0) = g(x)$$

where $g(x) = 1$ inside $(-1,1)$ and $g(x) = 0$ elsewhere.

(3) Suppose that $u(x,t)$ $(x \in \mathbb{R}, t > 0)$ solves the heat equation $\partial_t u = \partial_{xx} u$. For a given $a > 0$, consider the function $v(x,t) = v(x+at,t)$. Find the partial differential equation solved by $v(x,t)$.

(4) **Bonus.** For each $\epsilon > 0$ let $u^{(\epsilon)}(x,t)$ be the solution to the initial value problem

$$\partial_t u^{(\epsilon)} - \partial_x u^{(\epsilon)} = \epsilon \partial_{xx} u^{(\epsilon)}$$
$$u(x,0)^{(\epsilon)} = u_0(x).$$

(a) Using the previous problem, and the formula for the solution of the heat equation in 1-d, find a formula for the solution $u^{(\epsilon)}(x,t)$.
(b) Using the formula from part a), find the limit of $u^{(\epsilon)}(x,t)$ as $\epsilon \to 0^+$. Why is this interesting?.

(5) **Bonus.** Consider the semigroup $S(t)$ of transformations of functions on $\mathbb{R}$ defined by the formula

$$(S(t)u)(x) := u(x+t).$$

Then, check the following identities[2] for any time t and any differentiable $u : \mathbb{R} \to \mathbb{R}$,

$$\text{a)}\quad \frac{d}{dx}(S(t)u) = (S(t)\frac{d}{dx}u).$$
$$\text{b)}\quad \frac{d}{dt}(S(t)u) = \frac{d}{dx}(S(t)u).$$
$$\text{c)}\quad \frac{d}{dt}\left[\int_0^t (S(t-s)u\, ds\right] = u.$$

[2]i.e. for any u and t, the expressions given on the left and right hand side give the same function on $\mathbb{R}$