**Math 534H**

**Homework IV**
(Due Thursday, April 2nd)

(1) Find a explicit expression for the solution to

$$\begin{cases} \partial_t u &= \partial_{xx} u + 10u \qquad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u(x,0) &= \cos(x) + \cos(3x) \end{cases}$$

*Hint: Note that for every $n$, $\partial_{xx}(\cos(nx)) + 10\cos(nx) = (10-n^2)u$, use the fact that the initial data is a sum of cosines.*

(2) Find a explicit expression for the solution to each of the following problems

$$\text{a)} \begin{cases} \partial_t u &= \partial_{xx} u + 7\sin(2x) + 2\cos(3x) \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u(x,0) &= \cos(x). \end{cases}$$

$$\text{b)} \begin{cases} \partial_t u &= \partial_{xx} u + \sin(4x) - 2\cos(5x) \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u(x,0) &= \sin(2x) - 23\cos(7x) \end{cases}$$

$$\text{c)} \begin{cases} \partial_t u &= \partial_{xx} u + \sin(x) + \sin(2x) + \sin(4x) \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u(x,0) &= \sin(x) - \cos(x) \end{cases}$$

*Hint: Think of the problems as linear systems of ODEs, remember variations of parameters?.*

(3) For every $n \in \mathbb{N}$, check that the complex valued function

$$E(x,t) = e^{inx - n^2 t}$$

is a solution to the heat equation, $(x,t) \in \mathbb{R}\times\mathbb{R}$.

(4) Consider the $2\pi$-periodic heat kernel, that is the function

$$H(x,t) = \frac{1}{\pi}\sum_{n=0}^{\infty} \mathrm{Re}\left(e^{inx-n^2t}\right).$$

(a) Check that $\partial_t H = \partial_{xx} H$.

(b) For $t > 0$ let[1] $S(t) : C[0,2\pi] \to C[0,2\pi]$ denote the linear operator that maps a function $u(x)$ to a new function $S(t)u$ defined by

$$(S(t)u)(x) = \int_0^{2\pi} H(x-y,t)u(y)\,dy.$$

Check that $u(x,t) = (S(t)u)(x)$ solves $\partial_t u = \partial_{xx} u$.

[1] $C[a,b]$ denote the set of all continuous functions in the interval $[0,2\pi]$.

(c) (Variation of parameters/Duhamel's formula) Given a function $f(x)$, check that the function

$$v(x,t) = \int_0^t (S(t-s)f)(x)\, ds$$

solves

$$\partial_t v = \partial_{xx} v + f(x).$$

(5) Let $u(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ solve

$$\partial_t u = \partial_{xx} u, \quad u(x,0) = u_0(x).$$
$$u(x,t) \text{ is } 2\pi\text{-periodic in } x.$$

Justify the formula

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^{2\pi} (u(x,t))^2\, dx\right) = -\int_0^{2\pi} (\partial_x u(x,t))^2\, dx$$

Conclude that $\int_0^{2\pi} u(x,t)^2\, dx$ is decreasing with time.

(6) (Bonus) ("Finite speed of propagation") Suppose that $u$ solves the **porous medium equation**,

$$\partial_t u = \partial_{xx}(u^2)$$

With $u(x,0) = u_0(x)$ a nonnegative function such that $u_0 \leq 1$ everywhere and $u_0(x) \equiv 0$ if $x \notin (-1,1)$. Find a function $R(t)$ so that

$$u(x,t) \equiv 0 \text{ outside } (-R(t), R(t)).$$

*Hint:* Use the comparison principle with $u$ and a well chosen special solution (also, compare this phenomenon with what is obtained in the second bonus problem).

(7) (Bonus) Generalize problem #1 to higher dimensions, so, $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ and such that $u(x_1, \ldots, x_d)$ is $2\pi$-periodic in each of its variables and $\partial_t u = \Delta u$. Show that

$$\frac{1}{2}\frac{d}{dt}\left(\int_{[0,1]^d} (u(x,t))^2\, dx\right) = -\int_{[0,1]^d} |\nabla u(x,t)|^2\, dx.$$

(8) (Bonus) ("Infinite speed of propagation") Consider $u$ a solution of the heat equation

$$\partial_t u = \partial_{xx} u \quad \text{if} \quad \mathbb{R} \times \mathbb{R}_+$$

where $u(x,0) = u_0(x)$, $u_0$ vanishes outside $(-1,1)$ and $u_0(x) > 0$ for every $x$ in $(-1,1)$.

(a) Check that no matter how small $t > 0$ is, we have $u(x,t) > 0$ for **every** $x \in \mathbb{R}$.

(b) Suppose you know further that $u_0(x) \geq 2$ everywhere in $(0, 1/2)$, then show the lower estimate

$$u(x,t) \geq \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad \forall\, t > 0, x > 0.$$

*Hint: For part a) note that if $h$ is a function which is strictly positive everywhere and $u_0 \geq 0$, then the integral*

$$\int_{\mathbb{R}} h(x) u_0(x)\, dx$$

*can only be zero if $u_0 = 0$ everywhere.*