Solutions for Homework #3

Problem 1. We compute directly: for given integers n, m, we have that

$$\begin{aligned}\partial_{x_1x_1} u(x_1, x_2) &= \partial_{x_1x_1} \left(\sin(2\pi n x_1)\sin(2\pi m x_2)\right)\\ &= \partial_{x_1x_1} \left(\sin(2\pi n x_1)\right)\sin(2\pi m x_2)\\ &= -(2\pi n)^2 \sin(2\pi n x_1)\sin(2\pi m x_2)\\ &= -(2\pi n)^2 u.\end{aligned}$$

The exact same computation but with x_2 yields

$$\partial_{x_2x_2} u(x_1, x_2) = -(2\pi m)^2 u$$

Therefore,

$$\Delta u = (-2\pi)^2(n^2 + m^2)u \text{ in } \Omega.$$

On the other hand, note that since n, m are integers, then $\sin(2\pi n x)$ and $\sin(2\pi m x)$ both vanish for $x = 0$ or $x = 1$. It follows that u must be zero whenever at least one of x_1, x_2 is equal to 0 and 1, but this is always the case in the boundary of the square $\Omega = [0, 1] \times [0, 1]$. Therefore the u above is always zero on $\partial\Omega$, regardless of n and m.

Finally, we observe that by denoting $u_{n,m}(x_1, x_2) = \sin(2\pi n x_1)\sin(2\pi m x_2)$ for $(n, m) \in \mathbb{Z}^2$ then we get $\Delta u_{n,m} = \lambda_{n,m} u$, with $\lambda_{n,m} = (-2\pi)^2(n^2 + m^2)$ i.e. an infinite list of "eigenvalues" $\lambda_{n,m}$ for the linear operator Δ, and we are done.

Problem 2. Last time in the 1-d case we were able to integrate by parts to get an integral identity that, when $\lambda \geq 0$, guaranteed that the solution had to be equal to zero everywhere. So, we are going to do exactly the same in this higher dimensional case, using the divergence theorem/Green's identity.

As before: we multiply both sides of the equation by the solution itself, and integrate over Ω. We get

$$\int_\Omega u\Delta u \, dx = \lambda \int_\Omega u^2 \, dx \geq 0.$$

On the other hand, we have

$$\int_\Omega u\Delta u \, dx = \int_{\partial\Omega} u\partial_n u \, d\sigma(x) - \int_\Omega |\nabla u|^2 \, dx = -\int_\Omega |\nabla u|^2 \, dx$$

where we used that $u \equiv 0$ on $\partial\Omega$ to get that the boundary integral was zero. In conclusion,

$$-\int_\Omega |\nabla u|^2 \, dx = \lambda \int_\Omega u^2 \, dx \geq 0$$

This means that the integral of $|\nabla u|^2$ over Ω must be zero, and since this integrand is nonnegative, that $|\nabla u|^2$ (and thus ∇u) must be zero everywhere in Ω as well, so u must be zero (since it follows it is a constant and we know it is zero on the boundary).

Problem 3. The problem says that v and u are related by

$$v(x) = u(Lx)$$

In coordinates, if $x = (x_1, x_2)$ then $Lx = (l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)$.

$$v(x_1, x_2) = u(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)$$

It is all now a matter of computing $v_{x_1x_1}$ and $v_{x_2x_2}$ by two succesive uses of the chain rule. First,

$$\begin{aligned}
\partial_{x_1} v &= l_{11}(\partial_{x_1} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2) + l_{21}(\partial_{x_2} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)\\
\partial_{x_2} v &= l_{12}(\partial_{x_1} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2) + l_{22}(\partial_{x_2} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)
\end{aligned}$$

Next,

$$\begin{aligned}
\partial_{x_1x_1} v = l_{11}&\left[l_{11}(\partial_{x_1x_1} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2) + l_{21}(\partial_{x_1x_2} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)\right] +\\
&+ l_{21}\left[l_{11}(\partial_{x_2x_1} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2) + l_{21}(\partial_{x_2x_2} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)\right]
\end{aligned}$$

and,

$$\begin{aligned}
\partial_{x_2x_2} v = l_{12}&\left[l_{12}(\partial_{x_1x_1} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2) + l_{22}(\partial_{x_1x_2} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)\right] +\\
&+ l_{22}\left[l_{12}(\partial_{x_2x_1} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2) + l_{22}(\partial_{x_2x_2} u)(l_{11}x_1 + l_{12}x_2, l_{21}x_1 + l_{22}x_2)\right]
\end{aligned}$$

Adding these up, and omitting (for the sake of brevity) the point where are evaluating the second derivatives of u, we arrive at

$$\begin{aligned}
\Delta v(x) = (l_{11}^2 + l_{12}^2)&\partial_{x_1x_1} u + (l_{11}l_{21} + l_{12}l_{22})\partial_{x_1x_2} u +\\
&+ (l_{21}l_{11} + l_{22}l_{12})\partial_{x_2x_1} u + (l_{21}^2 + l_{22}^2)\partial_{x_2x_2} u.
\end{aligned}$$

Problem 4. L being a rotation means that for some θ we have

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Substituting these in the formula for Δv obtained in the previous problem,

$$\begin{aligned}
\Delta v(x) &= (\cos(\theta)^2 + \sin(\theta)^2)\partial_{x_1x_1} u + (\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta))\partial_{x_1x_2} u +\\
&\quad + (\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta))\partial_{x_2x_1} u + (\sin(\theta)^2 + \cos(\theta)^2)\partial_{x_2x_2} u\\
&= (1)\partial_{x_1x_1} u + (0)\partial_{x_1x_2} u + (0)\partial_{x_2x_1} u + (1)\partial_{x_2x_2} u\\
&= \partial_{x_1x_1} u + \partial_{x_2x_2} u.
\end{aligned}$$

Since the second derivatives of u are being evaluated at the point Lx, we obtain the formula.

Problem 5. We use the chain and Leibniz rule to compute the gradient of the composition of u with the different functions, and the Leibniz rule to compute the divervence. So,

(a) $\Delta(u^3) = \text{div}\left(3u^2\nabla u\right) = 3u^2\Delta u + 6u|\nabla u|^2$
(b) $\Delta(\sin(u)) = \text{div}\left(\cos(u)\nabla u\right) = \cos(u)\Delta u - \sin(u)|\nabla u|^2$
(c) $\Delta(e^u) = \text{div}\left(e^u\nabla u\right) = e^u\Delta u + e^u|\nabla u|^2$

Recall: by the chain rule we have that if $u : \Omega \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$, then $\nabla\phi(u) = \phi'(u)\nabla u$ and by the Leibniz rule we also have for any $\psi : \mathbb{R} \to \mathbb{R}$, $\text{div}(\psi(u)\nabla u) = \psi(u)\Delta u + \psi'(u)|\nabla u|^2$. Combining these two formulas with $\psi = \phi'$, we get

$$\Delta(\phi(u)) = \text{div}\left(\phi'(u)\nabla u\right) = \phi'(u)\Delta u + \phi''(u)|\nabla u|^2.$$

As we wanted.

Problem 6. This one has an short solution if you know what to use: from the previous problem we learned how to get an inequality for $\Delta(\phi(u))$ (using also that $\Delta u = 0$, of course). We get

$$\begin{aligned}\Delta(\phi(u)) &= \phi'(u)\Delta u + \phi''(u)|\nabla u|^2 \\ &= \phi''(u)|\nabla u|^2 \geq 0\end{aligned}$$

where we used that $\phi'' \geq 0$. Since $\Delta(\phi(u)) \geq 0$, Problem #3 from Homework II says that $v = \phi(u)$ is subharmonic, i.e. that

$$v(x_0) \leq \frac{1}{2\pi r}\int_{\partial D_r(x_0)} v(x)\, d\sigma(x)$$

and

$$v(x_0) \leq \frac{1}{\pi r^2}\int_{D_r(x_0)} v(x)\, dx.$$

Problem 7.

(a) Using the divergence theorem/Green's identity, we know that

$$\int_\Omega \nabla\phi\cdot\nabla u\, dx = \int_{\partial\Omega}\phi\partial_n u\, d\sigma(x) - \int_\Omega \phi\Delta u\, dx = 0$$

The first integral being zero due to $\phi = 0$ on $\partial\Omega$ and the second being zero since $\Delta u = 0$ everywhere in Ω.

(b) This is simply a quadratic expression, in fact, for any $x \in \Omega$,

$$|\nabla(u+\phi)(x)|^2 = |\nabla u(x) + \nabla\phi(x)|^2 = |\nabla u(x)|^2 + 2\nabla u(x)\cdot\nabla\phi(x) + |\nabla\phi(x)|^2$$

Integrating this expression in Ω we get the formula in (b).

(c) If $u = v$ on $\partial\Omega$, then $\phi = u - v$ is a function which vanishes on $\partial\Omega$, and $v = u + \phi$. Then, by part (b),

$$\begin{aligned}\int_\Omega |\nabla v|^2 &= \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega \nabla u\cdot\nabla\phi\, dx + \int_\Omega |\nabla\phi(x)|^2\, dx \\ &= \int_\Omega |\nabla u|^2\, dx + \int_\Omega |\nabla\phi|^2\, dx \\ &\geq \int_\Omega |\nabla u|^2\, dx\end{aligned}$$

where we used the harmonicity of u and (a) to see that the middle integral on the right must be zero. This gives (c).

(d) In the last formula in (c), the only way the $\geq$ can be an equality is if the integral of $|\nabla\phi|^2$ is zero, which can only happen if ϕ itself is zero ($\nabla\phi$ would need to be zero, so ϕ is a constant, and since it is zero on $\partial\Omega$, this constant is zero). But $\phi \equiv 0$ is the same $u \equiv v$, which gives (d).