Math 534H

Homework III

(Due Thursday, February 26th)

(1) Consider the square $\Omega = \{(x_1, x_2) \mid 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}$ and for every pair of numbers $n, m \in \mathbb{Z}$, the functions

$$u(x_1, x_2) = \sin(2\pi n x_1)\sin(2\pi m x_2).$$

Compute $\Delta u(x_1, x_2)$. Then, say whether there is a finite or infinite number of number λ's for which one can find a function $u : \Omega \to \mathbb{R}$ -different from the zero function- solving

$$\begin{cases} \Delta u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{cases}$$

(Meaning, if there are only finitely many, explain why. Otherwise, provide an infinite list of such λ's with their respective u's. The equation above is known as the *Helmholtz equation*, we'll see it again when we talk about the wave equation and *Schrödinger*'s equation)

(2) **The Nefarious Return Of Problem #6 From Homework II**. In Problem # 2 above, show that if $\lambda \geq 0$ then the only possible solution to the problem is $u \equiv 0$.

Hint: Following a 2-d analogue of the old hint, multiply both sides of the equation by u, integrate over Ω and use one of the Green identities.

(3) Suppose that L is a linear transformation of the plane onto itself (so, L is given by (l_{ij}), a 2×2 matrix). Let $u : \mathbb{R}^2 \to \mathbb{R}$. If v is a new function given by

$$v(x) = u(Lx),$$

then use the chain rule to write down $\partial_{x_i x_j} u$ in terms of an expression combining the second partial derivatives of u and the matrix (l_{ij}).

(4) In the previous problem, suppose that L is a rotation (so that the (l_{ij}) have a special form). Check that in this case

$$\Delta v(x) = (\Delta u)(Lx), \quad \forall\, x \in \mathbb{R}^2.$$

(5) (a) Use the Leibniz and chain rules to check the following identities (u is some unknown function)

$$\Delta u^3 = 6u|\nabla u|^2 + 3u^2 \Delta u,$$
$$\Delta(\sin(u)) = -\sin(u)|\nabla u|^2 + \cos(u)\Delta u,$$
$$\Delta(e^u) = e^u|\nabla u|^2 + e^u \Delta u.$$

(b) Based on the previous computations, guess (and check!) a formula for

$$\Delta\phi(u),$$

in terms of $|\nabla u|, \Delta u, \phi'(u)$ and $\phi''(u)$.

(6) Suppose that $\Delta u = 0$ in some domain $\Omega \subset \mathbb{R}^2$, and let $\phi(t)$ be a twice-differentiable, *convex* function of $t \in \mathbb{R}$ (i.e. $\phi''(t) \geq 0$). Then, prove that *for any* disc $D_r(x_0)$ whose closure is contained in Ω we have

$$\phi(u(x_0)) \leq \frac{1}{2\pi r} \int_{D_r(x_0)} \phi(u(x))\,dx.$$

Hint: Use the calculation from the previous problem and problem #3(a) from Homework II.

(7) Let Ω be a domain in $\mathbb{R}^2$.

(a) Check that if $u : \Omega \to \mathbb{R}$ is harmonic, then

$$\int_\Omega (\nabla u, \nabla \phi)\,dx = 0$$

for any other function ϕ which is zero on $\partial\Omega$.
Hint: Use the divergence theorem (or the Green identities) with vector field $\phi\nabla u$, and use the fact that this vector field is always zero on $\partial\Omega$.

(b) Show that

$$\int_\Omega |\nabla(u+\phi)|^2\,dx = \int_\Omega |\nabla u|^2\,x + 2\int_\Omega (\nabla u, \nabla\phi)\,dx + \int_\Omega |\nabla\phi|^2\,dx$$

(c) Using (a) and (b) show that if $u = v$ on $\partial\Omega$ and u is harmonic in Ω then

$$\int_\Omega |\nabla u|^2\,dx \leq \int_\Omega |\nabla v|^2\,dx.$$

(d) Let u and v as before, but further suppose that you have the equality

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega |\nabla v|^2\,dx.$$

Conclude in this case that $u = v$.
Remark: Note that none of the above used in any way the two dimensions, and each of the steps are exactly the same if we have instead a harmonic function $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^d$, $d \geq 1$.

(8) **Bonus.** Go on the internet and check out a short video called "Möbius transformations revealed" by Douglas N. Arnold and Jonathan Rogness (yes, just watch the video, that's it!).

(9) **Bonus.** (Do Problem #8 first). The *inversion* of a point $x \in \mathbb{R}^2$ ($x \neq 0$) with respect to a circle of radius r centered at the origin 0 is the point $T(x)$ given by

$$T(x) = \frac{r^2 x}{|x|^2}$$

If $x = (x_1, x_2)$ then in coordinates we have

$$T(x_1, x_2) = \left(\frac{r^2 x_1}{x_1^2 + x_2^2}, \frac{r^2 x_2}{x_1^2 + x_2^2}\right)$$

Check the following

(a) If $r = 1$, then $T(x)$ is its own inverse, meaning that for any $x \neq 0$, $T(T(x)) = x$.

(b) (Again take $r = 1$) Suppose Ω_1 and Ω_2 are two domains on the plane and that

$$0 \notin \Omega_1,\;\; T(\Omega_1) = \Omega_2$$

Given a function $u : \Omega_2 \to \mathbb{R}$, define a new function $v : \Omega_1 \to \mathbb{R}$ by composing u and T,

$$v(x) = u(T(x))$$

By succesive uses of the chain rule, compute the partial derivatives $\partial_{x_1 x_1} v$ and $\partial_{x_2 x_2} v$ in terms of the partial derivatives of u, and the map T (you may want to do the calculation first for $r = 1$ since thats easier).

(c) Assume that $\Omega_1 = \Omega_2 = \mathbb{R}^2 \setminus \{0\}$ and $u(x) = \log(|x|)$. What is $v(x)$ in this case?. Also, compute Δv. *Hint: Use the radial formula for the Laplacian.*

(d) Assume $\Delta u = 0$, what can you say about Δv?.

(10) **Bonus.** Given two harmonic functions u and v in Ω, let $w(x) := \max\{u(x), v(x)\}$. Show that w is always subharmonic in Ω.

Hint: Note that in general this function may fail to have a derivative!, so you cannot simply try arguing that $\Delta w \geq 0$ *since* w *might not eve have second-order derivatives.*