Solutions for Homework #2

Problem 1.

Solution. We know that the function $V_1(x) = \frac{1}{|x|}$ is harmonic away from $x = 0$ and that the constant function $V_0(x) = 1$ is harmonic everywhere. So any linear combination of them will be in particular harmonic in the ring $1 \leq |x| \leq 2$. Thus, we seek for a solution to have the form

$$u(x) = A\frac{1}{|x|} + B.$$

Since the $u = 5$ when $|x| = 1$ and $u = 2$ when $|x| = 2$, we get the equations

$$5 = A + B, \quad 2 = \frac{A}{2} + B$$

Solving for A and B we get $A = 6, B = -1$ therefore the desired function is

$$u(x) = \frac{6}{|x|} - 1.$$

Problem 2.

Solution. **a).** First, let us compute the integrals

$$\int_{D_r(0)} x_1^2 \, dx, \int_{D_r(0)} x_2^2 \, dx, \int_{D_r(0)} x_1 x_2 \, dx.$$

By symmetry, the first two integrals two each other (which can be seen by a rotation by $\pi/2$), also by symmetry, the second integral must be zero (which can be seen by reflection with respect to any of the coordinate axes, using the fact that the function is odd in each of the two variables).

Since the first two integrals are the same, we can do the following trick, write

$$\int_{D_r(0)} x_1^2 \, dx = \frac{1}{2}\left(\int_{D_r(0)} x_1^2 \, dx + \int_{D_r(0)} x_2^2 \, dx\right) = \frac{1}{2}\int_{D_r(0)} x_1^2 + x_2^2 \, dx$$

The advantage being that the integral of $x_1^2 + x_2^2$ over the disc is easy to compute using polar coordinates:

$$\frac{1}{2}\int_{D_r(0)} x_1^2 + x_2^2 \, dx = \frac{1}{2}\int_0^r 2\pi\rho^2\rho \, d\rho = \pi\int_0^r \rho^3 \, d\rho = \frac{\pi}{4}r^4.$$

In conclusion,

$$\int_{D_r(0)} x_1^2 \, dx = \int_{D_r(0)} x_2^2 \, dx = \frac{\pi}{4}r^4, \quad \int_{D_r(0)} x_1 x_2 \, dx = 0.$$

Then, by linearity

$$\frac{1}{\pi r^2}\int_{D_r(0)} Ax_1^2 + Bx_1x_2 + Cx_2^2 \, dx = \frac{1}{\pi r^2}\left(A\frac{\pi}{4}r^4 + B\frac{\pi}{4}r^4\right) =$$

$$= \frac{r^2}{4}(A + C)$$

which gives the average over $D_r(0)$.

b). If $u(x) = Ax_1^2 + Bx_1x_2 + Cx_2^2$ and $x_0 = (x_{01}, x_{02})$, then

$$\begin{aligned}\tilde{u}(x) = u(x - x_0) &= A(x_1 - x_{01})^2 + B(x_1 - x_{01})(x_2 - x_{02}) + C(x_2 - x_{02})^2 \\ &= Ax_1^2 - 2Ax_{01}x_1 + Ax_{01}^2 + Bx_1x_2 - Bx_{02}x_1 - Bx_{01}x_2 + Bx_{01}x_{02} \\ &\quad + Cx_2^2 - 2Cx_{02}x_1 + Cx_{02}^2\end{aligned}$$

Arranging the terms, and noting that $u(x_0) = Ax_{01}^2 + Bx_{01}x_{02} + Cx_{02}^2$, we get

$$\tilde{u}(x) = \left(Ax_1^2 + Bx_1x_2 + Cx_2^2\right) - (2Ax_{01}x_1 + Bx_{02}x_1 + Bx_{01}x_2 + 2Cx_{02}x_1) + u(x_0).$$

The first parenthesis contains a term whose integral over D_r was computed in part a), the term in the second parenthesis is a sum of linear functions, thus its integral over D_r is zero. The third term is the constant $u(x_0)$, so its integral over D_r is simply $\pi r^2 u(x_0)$. Adding these up, we get

$$\frac{1}{\pi r^2}\int_{D_r(0)} \tilde{u}(x)\, dx = \frac{r^2}{4}(A + C) + u(x_0).$$

On the other hand, doing the change of variables $y = x - x_0$, it follows that

$$\frac{1}{\pi r^2}\int_{D_r(x_0)} u(x)\, dx = \frac{1}{\pi r^2}\int_{D_r(0)} \tilde{u}(y)\, dy.$$

In conclusion

$$\frac{1}{\pi r^2}\int_{D_r(x_0)} u(x)\, dx = \frac{r^2}{4}(A + C) + u(x_0).$$

c). From part b), we have immediately that if $A + C = 0$, then

$$\frac{1}{\pi r^2}\int_{D_r(x_0)} u(x)\, dx = u(x_0).$$

for every radius r and every point x_0.

Problem 3.

(a) Following the proof of the mean value property when $\Delta u = 0$, we fix $x_0 \in \Omega$ and consider the following function of r

$$\phi(r) = \frac{1}{\text{Area}(\partial B_r)}\int_{\partial B_r(x_0)} u(x)\, d\sigma(x)$$

which is defined when r is smaller than the distance from x_0 to $\partial\Omega$. Since $\Delta u \geq 0$ everywhere, the formula for ϕ' obtained in the original proof gives us $\phi'(r) \geq 0$. In particular,

$$\phi(r) \geq \phi(0) = u(x_0).$$

Which is just another way of writing

$$u(x_0) \leq \frac{1}{\text{Area}(\partial B_r)}\int_{\partial B_r(x_0)} u(x)\, d\sigma(x)$$

Integrating this expression in r (as in the proof of the mean value property), we also get

$$u(x_0) \leq \frac{1}{\text{Vol}(B_r)}\int_{B_r(x_0)} u(x)\, dx.$$

Problem 4.

Solution. There is no contradiction to the maximum principle because the domain Ω is not connected.

Problem 5.

Solution. Let us use the formula for the Laplacian in polar coordinates:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Then, since $u(r,\theta) = r^{1/2}\cos(\theta/2)$ we get

$$\begin{aligned} u_r &= \frac{1}{2}r^{-1/2}\cos(\theta/2), \\ u_{rr} &= -\frac{1}{4}r^{-3/2}\cos(\theta/2), \\ u_{\theta\theta} &= -\frac{1}{4}r^{1/2}\cos(\theta/2). \end{aligned}$$

Then, combining all the terms,

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= -\frac{1}{4}r^{-3/2}\cos(\theta/2) + \frac{1}{r}\frac{1}{2}r^{-1/2}\cos(\theta/2) - \frac{1}{r^2}r^{1/2}\frac{1}{4}\cos(\theta/2), \\ &= \frac{1}{4}r^{-3/2}\cos(\theta/2) - \frac{1}{4}r^{-3/2}\cos(\theta/2), \\ &= 0. \end{aligned}$$

Problem 6. We are going to show that if $w(\theta)$ is **periodic** or if $w(0) = w(2\pi) = 0$ and $w'' = \lambda w$ with $\lambda \geq 0$ then $w \equiv 0$.

Multiply both sides of the differential equation by $w(\theta)$ and integrate from 0 to 2π to get

$$\int_0^{2\pi} w''(\theta)w(\theta)d\,\theta = \lambda \int_0^{2\pi} w(\theta)^2\,d\theta.$$

Integrating by parts on the left, we get

$$(w''(\theta)w(\theta))\,|_0^{2\pi} - \int_0^{2\pi}(w'(\theta))^2\,d\theta = \lambda\int_0^{2\pi} w(\theta)^2\,d\theta.$$

w being either periodic or being zero at 0 and 2π both guarantees that $(w''(\theta)w(\theta))\,|_0^{2\pi} = 0$, so in either case we get the relation

$$-\int_0^{2\pi}(w'(\theta))^2\,d\theta = \lambda\int_0^{2\pi} w(\theta)^2\,d\theta.$$

But both integrands are nonnegative, and since $\lambda \geq 0$ the only way the above equality can hold is when both integrals are zero, but then we conclude that $w \equiv 0$, as we wanted.

Problem 7.

a). No, $u(0,0)$ cannot be larger than 0, in fact, $u \leq -1/2$ at the origin.

(to see that at least $u \leq 0$) The function $v(x) = x_1^2 - 2x_2^2$ is superharmonic in D_1 and $u \leq v$ on ∂D_1, so the comparison principle says that $u \leq v$ everywhere in D_1, and in particular

$$u(0,0) \leq v(0,0) = 0.$$

We can get a better bound as follows: thanks to the mean value theorem and the continuity of u on $\partial \bar{D}_1$,

$$u(0,0) = \frac{1}{2\pi} \int_{\partial D_1} u(x)\, d\sigma(x)$$

By hypothesis, $u(x) \leq x_1^2 - 2x_2^2$ on the boundary, therefore,

$$\begin{aligned} u(0,0) &\leq \frac{1}{2\pi} \int_{\partial D_1} x_1^2 - 2x_2^2\, d\sigma(x), \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta)^2 - 2\sin(\theta)^2\, d\theta, \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta)^2\, d\theta - \frac{1}{\pi} \int_0^{2\pi} \sin(\theta)^2\, d\theta. \end{aligned}$$

Since,

$$\int_0^{2\pi} \cos(\theta)^2\, d\theta = \int_0^{2\pi} \sin(\theta)^2\, d\theta = \pi$$

it follows that

$$u(0,0) \leq -1/2.$$

Therefore, such a u cannot ever be bigger than $-1/2$, and this is sharp, just consider the case where $g(x)$ is not just less but exactly equal to $x_1^2 - 2x_2^2$, in which case the above argument says that $u(0,0) = -1/2$.

b). Note that $(1/\sqrt{2}, 1/\sqrt{2})$ lies on ∂D_1, since there $u(x) = g(x) \leq x_1^2 - 2x_2^2$ we get, by substitution,

$$u(1/\sqrt{2}, 1/\sqrt{2}) \leq 1/2 - 1 = -1/2.$$

Problem 8.

(a) Since we have a disc with constant boundary conditions, we try looking for a radially symmetric solution. Since the right hand side is constant, it makes sense looking for a solution with constant second derivatives, i.e. a second order polynomial. That leaves us one choice: linear combinations of $x_1^2 + x_2^2$ and constants.

Then, note that $\Delta(|x|^2) = \Delta(x_1^2 + x_2^2) = 2$. Therefore, we look for a solution of the form,

$$u(x) = A - \frac{1}{2}(x_1^2 + x_2^2)$$

for some constnat A. Since we want $u(x) = 0$ when $|x| = r$, $A - r^2/2 = 0$, thus $A = r^2/2$, so the solution is

$$u(x) = \frac{r^2}{2} - \frac{1}{2}(x_1^2 + x_2^2).$$

(b) Since $\Delta u = 1$, u cannot be a constant. On the other hand, $\Delta u < 0$, so that u is superharmonic (alternatively, $-u$ has a nonnegative Laplacian, so its must be subharmonic). By Problem 3, it follows that u cannot have any minimum points in the interior of Ω, and in particular the minimum must happen on the boundary. Since u is always zero on the boundary, we conclude both that the minimum of u is 0 and that $u > 0$ inside Ω.

(c) Remember how we saw the partial derivatives of a harmonic function are harmonic? The same idea applies here: we compute the Laplacian of the partial derivatives and reverse the order of differentiation, since the Laplacian of u is constant (even if not zero), the Laplacian of the derivatives of u is zero. In formulas:

$$\Delta(u_{x_1}) = u_{x_1x_1x_1} + u_{x_1x_2x_2} = u_{x_1x_1x_1} + u_{x_2x_2x_1} = (\Delta u)_{x_1} = (-1)_{x_1} = 0.$$
$$\Delta(u_{x_2}) = u_{x_2x_1x_1} + u_{x_2x_2x_2} = u_{x_1x_1x_2} + u_{x_2x_2x_2} = (\Delta u)_{x_2} = (-1)_{x_2} = 0.$$

So, u_{x_1}, u_{x_2} are harmonic. Since we are interested in their squares (and we intend to use again the solution to Problem #3 [1]), we compute $\Delta(u_{x_1}^2), \Delta(u_{x_2}^2)$: a simple use of the chain and Leibniz rules give us that

$$\Delta(u_{x_1}^2) = 2|\nabla u_{x_1}|^2 + 2u_{x_1}\Delta(u_{x_1}) = 2|\nabla u_{x_1}|^2 \geq 0$$
$$\Delta(u_{x_2}^2) = 2|\nabla u_{x_2}|^2 + 2u_{x_2}\Delta(u_{x_2}) = 2|\nabla u_{x_2}|^2 \geq 0$$

Plus, since $|\nabla u|^2 = u_{x_1}^2 + u_{x_2}^2$, we also see that

$$\Delta(|\nabla u|^2) = 2(|\nabla u_{x_1}|^2 + |\nabla u_{x_2}|^2) \geq 0.$$

So $u_{x_1}^2, u_{x_2}^2$ and $|\nabla u|^2$ all have nonnegative Laplacians (i.e. they are subharmonic), in which case (by Problem #3) we know that they must achieve their maximum on the $\partial\Omega$.

Problem 9. Fix x_0 and write the integral in polar coordinates centered at the point x_0, we get

$$\int_{\mathbb{R}^2} u(x)\eta(x - x_0)\, dx = \int_0^\infty \int_{\partial D_1(0)} u(x_0 + rx)\eta(rx) r\, d\sigma(x)\, dr$$

Moreover, since $|x| = 1$ and η is zero outside $B_1(x)$, we see that $\eta(rx) = 0$ for $r \geq 1$, so that the above is in fact equal to

$$\int_0^1 \int_{\partial D_1(0)} u(x_0 + rx)\eta(rx) r\, d\sigma(x)\, dr$$

Since $\eta(rx)$ has the same value for any $x \in \partial D_1(0)$, we can simply call this common value $\eta(r)$ and take out of the integral over ∂D_1, so the integral is equal to the integral

$$\int_0^1 \eta(r) r \int_{\partial D_1(0)} u(x_0 + rx)\, d\sigma(x)\, dr \tag{0.1}$$

Now, since u is harmonic, the mean value property says that

$$u(x_0) = \frac{1}{2\pi r} \int_{\partial D_r(x_0)} u(x)\, d\sigma(x) = \frac{1}{2\pi} \int_{\partial D_1(0)} u(x_0 + rx)\, d\sigma(x)$$

[1] the reasoning being "how many tricks do we know that allow us to show a minimum/maximum must happen on the boundary? oh! right, the mean value property for harmonic functions and the version for subharmonic/superharmonic functions"

(the last identity being by the substitution $x \to x_0 + rx$ which turns the integral over $\partial D_r(x_0)$ into one over $\partial D_1(0)$)

Substituting this last formula in (0.1), we see that (0.1) is equal to

$$\int_0^1 \eta(r) r u(x_0) 2\pi \; dr = u(x_0) \int_0^1 \eta(r) r 2\pi \; dr$$

Finally, we observe that (again, using polar coordinates, this time about 0), that

$$\int_0^1 \eta(r) r 2\pi \; dr = \int_{B_1} \eta(x) \; dx = 1$$

In conclusion, the original integral is equal to $u(x_0)$.

Problem 10.

We compute the partial derivatives of P one by one, and simplifying the expression for the second order derivatives as much as it seems reasonable. First, note that

$$\partial_{x_1} P = \frac{1}{\pi} \partial_{x_1} \left(\frac{x_2}{x_1^2 + x_2^2} \right) = -\frac{2}{\pi} \frac{x_1 x_2}{(x_1^2 + x_2^2)^2}$$

$$\begin{aligned} \partial_{x_1 x_1} P = -\frac{2}{\pi} \partial_{x_1} \left(\frac{x_1 x_2}{(x_1^2 + x_2^2)^2} \right) &= -\frac{2}{\pi} \left(\frac{x_2}{(x_1^2 + x_2^2)^2} - 4 \frac{x_1^2 x_2}{(x_1^2 + x_2^2)^3} \right) \\ &= -\frac{2}{\pi} \left(\frac{x_2(x_1^2 + x_2^2) - 4x_1^2 x_2}{(x_1^2 + x_2^2)^3} \right) \\ &= -\frac{2}{\pi} \left(\frac{-3x_1^2 x_2 + x_2^3}{(x_1^2 + x_2^2)^3} \right). \end{aligned}$$

As for $\partial_{x_2} P$

$$\begin{aligned} \partial_{x_2} P = \frac{1}{\pi} \partial_{x_2} \left(\frac{x_2}{x_1^2 + x_2^2} \right) &= \frac{1}{\pi} \left(\frac{1}{x_1^2 + x_2^2} - \frac{2x_2^2}{(x_1^2 + x_2^2)^2} \right) \\ &= \frac{1}{\pi} \left(\frac{x_1^2 + x_2^2}{(x_1^2 + x_2^2)^2} - \frac{2x_2^2}{(x_1^2 + x_2^2)^2} \right) \\ &= \frac{1}{\pi} \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \partial_{x_2 x_2} P = \frac{1}{\pi} \partial_{x_2} \left(\frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} \right) &= -\frac{1}{\pi} \left(\frac{-2x_2}{(x_1^2 + x_2^2)^2} - 2 \frac{(x_1^2 - x_2^2) 2x_2}{(x_1^2 + x_2^2)^3} \right) \\ &= \frac{2}{\pi} \left(\frac{-x_2(x_1^2 + x_2^2) - 2x_2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^3} \right) \\ &= \frac{2}{\pi} \left(\frac{-3x_1^2 x_2 + x_2^3}{(x_1^2 + x_2^2)^3} \right). \end{aligned}$$

From the formulas for $\partial_{x_1 x_1} P$ and $\partial_{x_2 x_2} P$ it is evident that they are the same except for the opposite sign, therefore

$$\partial_{x_1 x_1} P + \partial_{x_2 x_2} P = 0$$