Math 534H

Homework I
(Due Tuesday, January 27th)

(1) Use separation of variables to solve the initial value problem:

$$u'(t) = u^2(t)$$
$$u(0) = 1$$

(2) Use separation of variables to solve the initial value problem:

$$u'(t) - 3u(t) = 0$$
$$u(0) = 1$$

(3) Use separation of variables to find **all** solutions to the ordinary differential equation:

$$u'(r) - \frac{2}{r}u(r) = 0$$

(4) Solve the following system of ordinary differential equations

$$\dot{x}(t) = -y(t)$$
$$\dot{y}(t) = x(t)$$

and write down a formula for the solution $(x(t), y(t))$ taking the initial value $(1, 3)$.

(5) For a given function $f(x,t)$ we consider the following list of equations

(a) $\partial_x f(x,t) = 0$
(b) $\partial_t f(x,t) = 0$
(c) $\partial_x f(x,t) + \partial_t f(x,t) = 0$
(d) $\partial^2_{xt} f(x,t) = 0$

Then,:
7.1) If $f(x,t) = \sin(x)$, which of a), b), c), d) is true?

7.2) if $f(x,t) = \sin(x-t)$, say which of a), b), c), d) is true?

7.3) if $f(x,t) = \sin(x) - \sin(t)$, say which of a), b), c), d) is true?

(6) Find **all** solutions to the second order ordinary differential equation:

$$u''(r) + \frac{2}{r}u'(r) = 0$$

(7) Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 7 & 2 \\ 0 & 5 & 9 \\ 0 & 0 & -1 \end{pmatrix}$$

(8) Let $u(x, y)$ be a continuous function of two variables, and for any $r > 0$ let D_r denote the interior of the disc with center 0 and radius r, that is

$$D_r = \{(x, y) \mid x^2 + y^2 < r^2\}$$

Then, show that no matter what the function u is, then we always have

$$\frac{1}{\text{Area}(D_r))} \int_{D_r} u(x, y)\, dxdy = \frac{1}{\text{Area}(D_1)} \int_{D_1} u(rx, ry)\, dxdy$$

(9) Suppose that the function $f(x, t)$ is given by

$$f(x, t) = g(x - 2t)$$

for some single-variable function g. Then, calculate and simplify the expression:

$$\partial_x f + \frac{1}{2}\partial_t f$$

(10) Compute the derivative of the following function

$$F(x) = \frac{1}{2} \int_{x-1}^{x+1} \sin(ky)\, dy$$

where k is some undetermined real number. Use the answer to compute the indefinite integral (aka anti-derivative, aka primitive function) of the function $G(x)$ defined by

$$G(x) = \frac{k}{2} \int_{x-1}^{x+1} \cos(ky)\, dy$$

(11) Consider the following two variable function

$$F(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \sin(y)\, dy$$

Compute $\partial_x F, \partial_t F$, $\partial_{xx} F$ and $\partial_{tt} F$. Do you find any relations between these derivatives?

(12) A homogeneous polynomial of degree two is a function $P(x, y)$ given by

$$P(x, y) = ax^2 + 2bxy + cy^2$$

for some numbers a, b and c. Then, find the dimension of the vector space of homogeneous polynomials of degree two which satisfy the relation

$$\partial^2_{xx} P + \partial^2_{yy} P = 0.$$