Math 534 Spring 2015
Midterm Solutions

(1) (a) Find a explicit solution u to the problem (in $\mathbb{R}^3$)

$$\begin{cases} \Delta u &= 0 \text{ in } B_2 \setminus B_1(0) \\ u &= 1 \text{ on } \partial B_1(0) \\ u &= 0 \text{ on } \partial B_2(0) \end{cases}$$

(b) Do the same for the slightly different problem (still in $\mathbb{R}^3$)

$$\begin{cases} \Delta u &= -6 \text{ in } B_2 \setminus B_1(0) \\ u &= 1 \text{ on } \partial B_1(0) \\ u &= 0 \text{ on } \partial B_2(0) \end{cases}$$

Solution. a) Since the problem is posed on a spherically symmetric domain, and the data is itself rotationally symmetric, we look for a radially symmeric solution u. We know that the constant function 1 and the function $|x|^{-1}$ are harmonic in our domain, so it makes sense to look for a solution which is a linear combination of these two, so we guess

$$u(x) = A + B|x|^{-1}$$

Substituting for the boundary values for $|x| = 1, 2$

$$|x| = 1, A + B = 1$$
$$|x| = 2, A + B/2 = 0$$

This system of equations is solved by the pair $A = -1, B = 2$. So, the solution is

$$u(x) = -1 + 2|x|^{-1}$$

b) Again by symmetry, we look for a function that depends only on $|x|$, since we want the Laplacian to be constant, we guess $u(x)$ should involve the quadratic function $|x|^2$. Now

$$\Delta(-|x|^2) = -6 \ \ \forall\, x$$

which is the right equation inside the domain but does not match the boundary values. Following the first case, we look for a solution of the form

$$u(x) = -|x|^2 + A + B|x|^{-1}$$

We check that it solves the equation in the domain (for any A and B)

$$\Delta u = \Delta(-|x|^2) + \Delta(A + B|x|^{-1}) = -6 + 0 = -6.$$

Now, we find A and B by using the boundary values we want at $|x| = 1, 2$,

$$|x| = 1, -1 + A + B = 1$$
$$|x| = 2, -4 + A + B/2 = 0$$

We get the system of equations $A + B = 2, 2A + B = 8$, which has $A = 6$ and $B = -4$ as solution. Then, the solution is given by

$$u(x) = -|x|^2 - 4\frac{1}{|x|} + 6.$$

(2) Consider a function $\phi : \mathbb{R} \to \mathbb{R}$ having derivatives of order 2. Let $u(x) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be given by

$$u(x) = \phi(|x|)$$

(a) Find a formula for Δu in terms of ϕ'', ϕ', $|x|$ and the dimension n.

(b) Use the first part of the problem to compute

$$\Delta(\log(|x|)) \text{ in } \mathbb{R}^2 \setminus \{0\}$$
$$\Delta(|x|^{-2}) \text{ in } \mathbb{R}^3 \setminus \{0\}$$
$$\Delta(|x|^{-2}) \text{ in } \mathbb{R}^4 \setminus \{0\}.$$

Hint: Recall that when $x \neq 0$, then $\nabla(|x|) = \frac{x}{|x|}$.

Solution. a) By the chain rule and the formula given by the hint

$$\nabla u(x) = \phi'(|x|)\nabla(|x|) = \phi'(|x|)\frac{x}{|x|}, \quad \text{when } x \neq 0.$$

Then, by the Leibniz rule (and the chain rule, again)

$$\begin{aligned}\Delta u(x) = \operatorname{div}\left(\phi'(|x|)\frac{x}{|x|}\right) &= \nabla(\phi'(|x|)) \cdot \frac{x}{|x|} + \phi'(|x|)\operatorname{div}\left(\frac{x}{|x|}\right) \\ &= \phi''(|x|)\frac{x}{|x|} \cdot \frac{x}{|x|} + \phi'(|x|)\left(\frac{1}{|x|}\operatorname{div}(x) + x \cdot \nabla(-\frac{1}{|x|})\right) \\ &= \phi''(|x|) + \phi'(|x|)\left(\frac{n}{|x|} - x \cdot (-\frac{x}{|x|^3})\right)\end{aligned}$$

Since in dimension n, $\operatorname{div}(x) = n$. Simplyfying, we get the formula we wanted for Δu:

$$\Delta u(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|).$$

b) If $\phi(t) = \log(t)$, then $\phi'(t) = t^{-1}, \phi''(t) = t^{-2}$, so for $n = 2$

$$\Delta(\log(|x|)) = |x|^{-2} + \frac{2-1}{|x|}\frac{1}{|x|} = |x|^{-2} - |x|^{-2} = 0 \quad \text{when } x \neq 0.$$

If $\phi(t) = t^{-2}$, then $\phi'(t) = -2t^{-3}, \phi''(t) = 6t^{-4}$, so for $n = 3$,

$$\Delta(|x|^{-2}) = 6|x|^{-4} + \frac{3-1}{|x|}(-2|x|^{-3}) = 6|x|^{-4} - 4|x|^{-4} = 2|x|^{-4} \quad \text{when } x \neq 0.$$

Finally, if again $\phi(t) = t^{-2}$, but $n = 4$, then

$$\Delta(|x|^{-2}) = 6|x|^{-4} + \frac{4-1}{|x|}(-2|x|^{-3}) = 6|x|^{-4} - 6|x|^{-4} = 0 \quad \text{when } x \neq 0.$$

So, the inverse squared of the modulus of x is not harmonic in $d = 3$, but it is so in $d = 4$!

(3) Consider the functions (as usual, $x = (x_1, x_2)$)

$$u(x) = (x_1 - 1)(x_2 - 1) + \frac{1}{7}(x_1^2 - x_2^2) \text{ in } \mathbb{R}^2.$$

$$h(x) = \begin{cases} \frac{5}{3\pi}(1 - |x|^2) & \text{in } D_1(0) \\ 0 & \text{outside } D_1(0). \end{cases}$$

Find the exact value of the integral

$$\int_{\mathbb{R}^2} u(x)h(x)\, dx$$

Hint: Note that $\int_{\mathbb{R}^2} h(x)\, dx = 1$.

Solution. We note that the functions $(x_1 - 1)(x_2 - 1) = x_1x_2 - x_1x_2 + 1$i is a sum of harmonic functions (a constant, a linear function, and x_1x_2 which is a second degree harmonic polynomial), thus itself harmonic. Also, $x_1^2 - x_2^2$ is also harmonic. Thus the function $u(x)$ is harmonic (in the entire plane).

On the other hand, $h(x)$ is a radially symmetric function with total integral 1. THUS, the mean value property applied to u (as done in one of the homework problems) shows that

$$\int_{\mathbb{R}^2} u(x)h(x)\, dx = u(0).$$

Since $u(0) = (0 - 1)(0 - 1)\frac{1}{7}(0^2 - 0^2) = 1$ we conclude that

$$\int_{\mathbb{R}^2} u(x)h(x)\, dx = 1.$$

(4) Solve (i.e. write as explicit a formula as you can) for the solution to the Dirichlet problem on

$$\begin{cases} \Delta u &= 0 \qquad \text{in } D_1(0) \\ u &= g(x) \qquad \text{on } \partial D_1(0) \end{cases}$$

when a) $g(x) = \sin(\theta)$ and b) $g(x) = 2\sin(2\theta) + 5\cos(3\theta)$.
Hint: Remember, Cartesian and polar coordinates are related by $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$.

Solution. When we derived the Poisson kernel for the disc we learned that -for any $m \in \mathbb{N}$- the functions given in polar coordinates by

$$r^m \sin(m\theta) \;\; r^m \cos(m\theta) \tag{$\star$}$$

are all harmonic. Now, the two Dirichlet boundary conditions given are linear combinations of these functions (with $m = 1, 3,$) evaluated at points where $r = 1$. Thus, the solution in both cases is the respective linear combinations of the functions ($\star$), and we get

a) $g(x) = \sin(\theta)$

$$u(x) = r\sin(\theta)$$

(note that this is a very simple function written in polar coordiantes, namely $u(x) = x_2$)

b) $g(x) = 2\sin(2\theta) + 5\cos(3\theta)$

$$u(x) = 2r^2\sin(2\theta) + 5r^3\cos(3\theta)$$

(this function turns out to be also a polynomial in x_1 and x_2, as can be seen by repeatedly applying the trigonometric identities for the sine and cosine of the sum of two angles and using as in **a)** that $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$.)

(5) A function u is harmonic in $D_1(0)$ and is such that

$$\int_{\partial D_1(0)} u(\nabla u \cdot n)\, d\sigma(x) = 0 \quad \text{and} \quad \int_{D_{1/2}(0)} u(x)\, dx = 0$$

Find u.

Solution. Since u is harmonic, the mean value property says that

$$u(0) = \frac{1}{\pi(1/4)} \int_{D_{1/2}(0)} u(x)\, dx = 0.$$

Since the integral on the right vanishes by assumption. So u is a harmonic function which is equal to zero at the origin.

On the other hand, the boundary integral is one the integrals appearing in the Green/integration by parts formulas, which suggests we use any of these formulas. For instance, using the divergence theorem (over the vector field $u\nabla u$), we see that

$$\begin{aligned} 0 = \int_{\partial D_1(0)} u(\nabla u \cdot n)\, d\sigma(x) &= \int_{D_1(0)} \mathrm{div}(u\nabla u)\, dx \\ &= \int_{D_1(0)} |\nabla u|^2 + u\Delta u\, dx \\ &= \int_{D_1(0)} |\nabla u|^2\, dx \end{aligned}$$

where we used the Leibniz rule and the assumption that $\Delta u = 0$ everywhere. Then

$$\int_{D_1(0)} |\nabla u|^2\, dx = 0$$

which means that $|\nabla u|^2$ is always zero in $D_1(0)$, in other words, u is a constant. Since we already saw that $u(0) = 0$ we see that u is in fact equal to the constant 0, so, we found u:

$$u(x) = 0, \;\; \forall\, x \in D_1(0).$$

(6) The nonlinear[1] boundary value problem (κ is some given constant)

$$\begin{cases} \Delta u &= \kappa e^u \text{ in } D_1(0) \\ u &= 0 \text{ on } \partial D_1(0) \end{cases}$$

is known as the **Yamabe problem**. Let u be the solution. Find the maximum value of u in $D_1(0)$ in the case when $\kappa \geq 0$ and find its minimum value in the case when $\kappa \leq 0$.

Solution. Whatever the solution u is, e^u is always a nonnegative function and by the equation for u it follows that Δu has always the same sign as the constant κ. Moreover, e^u is never zero, so in any case $\Delta u \neq 0$ which means that u cannot be a constant.

If $\kappa \geq 0$, then u is subharmonic and by the maximum principle it follows that it cannot achieve its maximum on the interior -we already know u cannot be a constant-. Therefore the only place where u can achieve its maximum is on the boundary, where it is always equal to zero. Then when $\kappa \geq 0$ we have $\max u = 0$.

If $\kappa \leq 0$, then u is superharmonic, in which case the maximum principle shows that u cannot achieve its minimum in the interior. Then, u achieves its minimum on the boundary, where it is always equal to zero. We conclude that $\min u = 0$ when $\kappa \leq 0$.

[1]this means that the f on the right hand side is itself a nonlinear function of the unknown u!

(7) Consider functions $u(x, y)$ and $v(x, y)$ (for this problem x, y are real numbers so (x, y) is a point in the plane). The pair u, v is said to satisfy the "Cauchy-Riemann conditions" if

$$\partial_x u = \partial_y v$$
$$\partial_y u = -\partial_x v.$$

(a) Check by direct computation that the these conditions are satisfied by the pair

$$u(x, y) = \text{Re}(x + iy)^2, \quad v(x, y) = \text{Im}(x + iy)^2$$

and by the pair

$$u(x, y) = \text{Re}(x + iy)^3, \quad v(x, y) = \text{Im}(x + iy)^3$$

(b) If u, v is any pairing satisfying these conditions, find Δu and Δv.

Hint for b): Keep in mind that $\partial_{xy} u = \partial_{yx} u$ for any u with continuous derivatives.

Solution. a) Computations for the first pair (recall that $u(x, y) = x^2 - y^2, v(x, y) = 2xy$)

$$\partial_x u = \partial_x (x^2 - y^2) = 2x$$
$$\partial_y u = \partial_y (x^2 - y^2) = -2y$$
$$\partial_x v = \partial_x (2xy) = 2y = -\partial_y u$$
$$\partial_y v = \partial_y (2xy) = 2x = \partial_x u$$

and we see that the first pair (u, v) satisfies the Cauchy-Riemman conditions

Computations for the second pair (note that $(x + iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$, so that $u(x, y) = x^3 - 3xy^2$ while $v(x, y) = 3x^2 y - y^3$)

$$\partial_x u = \partial_x (x^3 - 3xy^2) = 3x^2 - 3y^2$$
$$\partial_y u = \partial_y (x^3 - 3xy^2) = -6xy$$
$$\partial_x v = \partial_x (3x^2 y - y^3) = 6xy = -\partial_y u$$
$$\partial_y v = \partial_y (3x^2 y - y^3) = 3x^2 - 3y^2 = \partial_x u$$

so the second pair (u, v) also satisfies the Cauchy-Riemann conditions.

b) We write Δu as $\partial_x(\partial_x u) + \partial_y(\partial_y u)$ and substitute the relations for the first derivatives of u and v, we get

$$\Delta u = \partial_x (\partial_y v) + \partial_y (-\partial_x v)$$
$$= \partial_{xy} v - \partial_{yx} v = 0$$

Therefore, $\Delta u = 0$. The same argument for Δv is

$$\Delta v = \partial_x (-\partial_y u) + \partial_y (\partial_x u)$$
$$= -\partial_{xy} u + \partial_{yx} v = 0$$

So $\Delta v = 0$ as well.

(8) To any function u in a domain Ω, we assing the following "energy"

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |x|^2u^2\, dx$$

Prove:

(a) For any two functions u and ϕ: $E(u+\phi) = E(u) + E(\phi) + \int_\Omega \nabla u \cdot \nabla\phi\, dx + \int_\Omega |x|^2 u\phi\, dx$

(b) Suppose that among all functions vanishing on $\partial\Omega$ there is one u that minimizes E: u vanishes on $\partial\Omega$ and $E(u) \leq E(v)$ for any other v vanishing on $\partial\Omega$. Show that u solves the equation

$$-\Delta u + |x|^2 u = 0$$

Solution. a) This is simply the quadratic formula followed by splitting the resulting integral:

$$\begin{aligned} E(u+\phi) &= \frac{1}{2}\int_\Omega |\nabla(u+\phi)|^2 + |x|^2(u+\phi)^2\, dx \\ &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + 2\nabla u\cdot\nabla\phi + |\nabla\phi|^2\right) + |x|^2(u^2 + 2u\phi + \phi^2)\, dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + |x|^2u^2\, dx + \frac{1}{2}\int_\Omega |\nabla\phi|^2 + |x|^2\phi^2\, dx + \int_\Omega \nabla u\cdot\nabla\phi\, dx + \int_\Omega |x|^2u\phi\, dx \\ &= E(u) + E(\phi) + \int_\Omega \nabla u\cdot\nabla\phi\, dx + \int_\Omega |x|^2u\phi\, dx \end{aligned}$$

b) Let ϕ be any function that vanishes on $\partial\Omega$, then for any $t \in \mathbb{R}$ the function $u + t\phi$ has the same values as u on $\partial\Omega$, thus by the assumption

$$E(u+t\phi) \geq E(u), \quad \forall\, t.$$

Using the formula from part **a)** with $t\phi$ as the function, the inequality above says that

$$E(u+t\phi) = E(u) + E(t\phi) + \int_\Omega \nabla u\cdot\nabla(t\phi)\, dx + \int_\Omega |x|^2u(t\phi)\, dx \geq E(u), \quad \forall\, t$$

So, the function of t of the right is always ≥ 0 (for all t) and is equal to 0 when $t = 0$ -that is, it achieves its minimum at $t = 0$. Its derivative with respect to t is

$$2tE(\phi) + \int_\Omega \nabla u\cdot\nabla\phi\, dx + \int_\Omega |x|^2u\phi\, dx$$

Since this must vanish for $t = 0$, we conclude: for any ϕ vanishing on $\partial\Omega$ we have

$$\int_\Omega \nabla u\cdot\nabla\phi\, dx + \int_\Omega |x|^2u\phi\, dx = 0$$

Now, integerating by parts, this means that

$$\begin{aligned} 0 = \int_\Omega \nabla u\cdot\nabla\phi\, dx + \int_\Omega |x|^2u\phi\, dx &= \int_\Omega (-\Delta u)\phi + |x|^2u\phi\, dx \\ &= \int_\Omega \phi(-\Delta u + |x|^2u)\, dx \end{aligned}$$

We conclude that the integral $\int_\Omega \phi(-\Delta u + |x|^2u)\, dx$ is zero for **every** function ϕ vanishing on $\partial\Omega$, which can only happen if

$$-\Delta u + |x|^2u = 0,$$

as we wanted.