Math 456: Mathematical Modeling

Tuesday, March 6th, 2018

Counting visiting times et cetera

Tuesday, March 20th, 2018

Today

Today we will see:

1. Where are we? A quick review of last few classes.
2. Some comments on Problem Set 4.
3. Proof of the Strong Markov Property.

Next class:

1. Counting how often a state is visited.
2. Statement of the convergence theorem.

Review

Last couple of classes

1. Transient and recurrent states.
2. Stopping times and exit distributions.
3. Calculus on graphs and the Laplacian.
4. The Strong Markov Property.
5. Probabilities of returns.

Review

Transient and recurrent states

In a Markov chain, we said first (informally) that a state x is **recurrent** if starting from x we are guaranteed to *eventually* return to x, and said to be **transient** if there is some chance of *never* returning to x.

In terms of the first visit time T_x, this is written as

$$x \text{ is said to be transient if } \mathbb{P}_x[T_x < \infty] < 1,$$
$$x \text{ is said to be recurrent if } \mathbb{P}_x[T_x < \infty] = 1.$$

$$\big(\text{recall that } T_x = \min\{n \geq 0 \mid X_n = x\}\ \big)$$

Review

Stopping times and exit distributions

For a set $A \subset \mathcal{S}$, the first arrival time T_A is an example of a **stopping time**.
We were interested in its exit distribution

$$G_y(x) = \mathbb{P}_x[X_{T_A} = y],$$

this defined for $y \in A$ and $x \in \mathcal{S}$.

Review

Calculus on graphs and the Laplacian

Going back to the exit distributions, we found that

$$\begin{aligned} \Delta G_y(x) &= 0 \text{ if } x \notin A, \\ G_y(x) &= 0 \text{ if } x \in A, x \neq y, \\ G_y(y) &= 1. \end{aligned}$$

This system of equations is always sufficient to determine all the exit probabilities $G_y(x)$!!

Review

Calculus on graphs and the Laplacian

Consider a Markov chain with state space $\mathcal{S}$ and transition probability $\mathrm{p}(x, y)$.

Given $f : \mathcal{S} \to \mathbb{R}$, we defined

$$\Delta f(x) = \sum_{y} (f(y) - f(x)) \mathrm{p}(x, y).$$

Theorem (Strong Markov Property)

Let X_n denote a Markov Chain. If T is a stopping time and $Y_n := X_{T+n}$, then Y_n is also a Markov chain. Moreover, the transition probability for Y_n is the same as the one for X_n.

Review

Probabilities of return

Using the Strong Markov Property, we were able to compute the probability of returning to x at least k times,

$$\mathbb{P}_x[T_x^{(k)} < \infty] = \rho_{xx}^k$$

where

$$\rho_{xy} = \mathbb{P}_x[T_y < \infty]$$

Today

Next

1. Some comments on Problem Set 4.
2. Proof of the Strong Markov Property.
3. Counting how often a state is visited.

The Strong Markov Property

Theorem (Strong Markov Property)

Let X_n denote a Markov Chain. If T is a stopping time and $Y_n := X_{T+n}$, then Y_n is also a Markov chain. Moreover, the transition probability for Y_n is the same as the one for X_n.

The Strong Markov Property

Theorem (Strong Markov Property)

Let X_n denote a Markov Chain. If T is a stopping time and $Y_n := X_{T+n}$, then Y_n is also a Markov chain. Moreover, the transition probability for Y_n is the same as the one for X_n.

What does this mean? Equivalently, it means that

$$\mathbb{P}[Y_{n+1} = \alpha_{n+1}, Y_n = \alpha_n, \ldots Y_1 = \alpha_1]$$

is equal to $\mathrm{p}(\alpha_n, \alpha_{n+1})\mathrm{p}(\alpha_{n-1}, \alpha_n) \ldots \mathrm{p}(\alpha_1, \alpha_2)\mathbb{P}(Y_1 = \alpha_1)$.

The Strong Markov Property

Proof

Let us analyze first $\mathbb{P}[X_{T+n} = x_n, \ldots, X_{T+1} = x_1, X_T = x]$.

To do this, we first note that

$$\mathbb{P}[X_{T+n} = x_n, \ldots, X_{T+1} = x_1, X_T = x]$$
$$= \sum_{t=1}^{\infty} \mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = x, T = t]$$

The Strong Markov Property

Proof

Fix $t \in \mathbb{N}$. Consider the sets

$$H_t = \{\bar{\alpha} = (\alpha_1, \dots, \alpha_t) \mid X_1 = \alpha_1, \dots, X_t = \alpha_t \Rightarrow T = t\}$$
$$H_{t,x} = \{\bar{\alpha} \in H_t \mid \alpha_t = x\}$$

In words: H_t is the possible trajectories for the chain, up to time t, for which the $T = t$; and $H_{t,x}$ are those trajectories up to time t, with $T = t$, ending in x.

The Strong Markov Property

Proof

In particular, this means that

$$\{X_T = x, T = t\} = \bigcup_{\bar{\alpha} \in H_{t,x}} \{X_1 = \alpha_1, X_2 = \alpha_2, \ldots, X_t = \alpha_t\}$$

The Strong Markov Property

Proof

Therefore, the probability

$$\mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = x, T = t]$$

is given by the sum of the probabilities

$$\mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = \alpha_t, \ldots, X_1 = \alpha_1]$$

The sum being over $\bar{\alpha} \in H_{t,x}$

The Strong Markov Property

Proof

Therefore, the probability

$$\mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = x, T = t]$$

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$$\mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = \alpha_t, \ldots, X_1 = \alpha_1]$$

for all $\bar{\alpha} \in H_{t,x}$

The Strong Markov Property

Proof

Now, we know that thanks to the Markov property,

$$\mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = \alpha_t, \ldots, X_1 = \alpha_1]$$

is equal to

$$\mathrm{p}(x, x_1)\mathrm{p}(x_1, x_2) \ldots \mathrm{p}(x_{n-1}, x_n)\mathbb{P}(X_t = \alpha_t, \ldots X_1 = \alpha_1)$$

The Strong Markov Property

Proof

Adding up the $\bar{\alpha}$'s we see that

$$\mathbb{P}[X_{t+n} = x_n, \ldots, X_{t+1} = x_1, X_t = x, T = t]$$

is equal to

$$\mathrm{p}(x, x_1)\mathrm{p}(x_1, x_2) \ldots \mathrm{p}(x_{n-1}, x_n)\mathbb{P}[X_t = x, T = t]$$

The Strong Markov Property

Proof

… adding up in t, and dividing by $\mathbb{P}(X_T = x)$ we see that

$$\mathbb{P}[X_{T+n} = x_n, \ldots, X_{T+1} = x_1 \mid X_T = x]$$

is equal to

$$\mathrm{p}(x, x_1)\mathrm{p}(x_1, x_2) \ldots \mathrm{p}(x_{n-1}, x_n)$$

and it follows that X_{T+n} is a Markov Chain with same transition probability as X_n.

Math 456: Mathematical Modeling

Tuesday, March 6th, 2018

Counting times, periodicity, and convergence

Thursday, March 22th, 2018

Last time

Visiting Times

Given a state $y \in S$, let

$$N(y) := \#\{n \geq 1 \mid X_n = y\}$$

That is, $N(y)$ is the random variable given as the number of times n for which the Markov chain is at the state y. In particular, if y is visited infinitely many times, then $N(y) = \infty$.

We are going to prove a few useful facts about $N(y)$.

Last time

Visiting Times

First, note that

$$\mathbb{P}_x(N(y) \geq 1) = 1 - \mathbb{P}_x(N(y) = 0) = \rho_{xy}$$

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Recall that $\rho_{xy} = \mathbb{P}_x(T_y < \infty)$, the probability of eventually visiting y starting from x.

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Let us computer the probabilities for the other values of $N(y)$!

Last time

Visiting Times

We showed that for any $k \in \mathbb{N}$

$$\mathbb{P}_x[N(y) \geq k] = \mathbb{P}_x[T_y^{(k)} < \infty] = \rho_{xy}\rho_{yy}^{k-1}.$$

From here, one can compute the distribution of $N(y)$

$$\begin{aligned}
\mathbb{P}_x[N(y) = k] &= \mathbb{P}_x[N(y) \geq k] - \mathbb{P}_x[N(y) = k+1] \\
&= \rho_{xy}\rho_{yy}^{k-1} - \rho_{xy}\rho_{yy}^{k} \\
&= \rho_{xy}\rho_{yy}^{k-1}(1 - \rho_{yy})
\end{aligned}$$

Visiting Times

There is a simple formula for the **expected** number of visits

Lemma

$$\mathbb{E}_x[N(y)] = \frac{\rho_{xy}}{1 - \rho_{yy}}$$

Visiting Times

We use the following general (and useful!) fact about the expectation of a real variable Y

$$\mathbb{E}[Y] = \sum_{k=1}^{\infty} \mathbb{P}(Y \geq k).$$

Why is this true?

Visiting Times

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Why is this true?
Let χ_A denote the random variable which is equal to 1 if the even happens, and 0 otherwise

$$\chi_A := \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{otherwise} \end{cases}$$

Visiting Times

If a random variable Y takes only the values $k = 1, 2, \ldots$, we may write it as

$$Y = \sum_{k=1}^{\infty} \chi_{\{Y \geq k\}}$$

In which case, taking expectation on both sides

$$\mathbb{E}[Y] = \sum_{k=1}^{\infty} \mathbb{E}[\chi_{\{Y \geq k\}}]$$

But, it is clear that

$$\mathbb{E}[\chi_A] = \mathbb{P}(A)$$

and we obtain the formula.

Visiting Times

Proof of the Lemma.

We use the formula for expectation we just obtained, it yields

$$\mathbb{E}_x[N(y)] = \sum_{k=1}^{\infty} \mathbb{P}_x(N(y) \geq k)$$
$$= \sum_{k=1}^{\infty} \rho_{xy}\rho_{yy}^{k-1}$$

Therefore, we have

$$\mathbb{E}_x[N(y)] = \rho_{xy} \sum_{k=0}^{\infty} \rho_{yy}^{k}$$

Visiting Times

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Visiting Times

Proof (continued).

Recall that if $t \in (0,1)$, then

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

Visiting Times

Proof (continued).

Recall that if $t \in (0,1)$, then

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We obtain, using the above with $t = \rho_{yy}$,

$$\mathbb{E}_x[N(y)] = \frac{\rho_{xy}}{1-\rho_{yy}} \qquad \text{if } \rho_{yy} < 1$$

$$\mathbb{E}_x[N(y)] = \infty \qquad \text{if } \rho_{yy} = 1$$

□

Visiting Times

A few takeaways from this formula: as long as $x \to y$, $\mathbb{E}_x[N(y)]$ gives us information about whether y is recurrent or not.

There is a completely different, formula for $\mathbb{E}_x[N(y)]$.

Visiting Times

Lemma

For any states x and y we have

$$\mathbb{E}_x[N(y)] = \sum_{n=1}^{\infty} \mathrm{p}^n(x, y)$$

Visiting Times

Proof.

Notice that we may write,

$$N(y) = \sum_{n=1}^{\infty} \chi_{\{X_n = y\}}$$

But

$$\mathbb{E}_x[\chi_{\{X_n = y\}}] = \mathrm{p}^n(x, y)$$

Therefore,

$$\mathbb{E}_x[N(y)] = \sum_{n=1}^{\infty} \mathrm{p}^n(x, y)$$

□

Visiting Times

Combining these two formulas tell us the following.
If y is a transient state, then for any state x

$$\sum_{n=1}^{\infty} \mathrm{p}^n(x,y) < \infty$$

and in particular, for a **transient** y the n-step transition probability from x to y goes to zero as n goes to infinity!

$$\lim_{n\to\infty} \mathrm{p}^n(x,y) = 0$$

Visiting Times

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Question:
Does $\mathrm{p}^n(x,y)$ have a limit as $n \to \infty$ for **recurrent** y?

Convergence Theorem

Theorem

Consider an irreducible, aperiodic chain, and let $\pi(y)$ *denote its stationary distribution.*

Then, for any $y \in S$*, we have*

$$\lim_{n\to\infty} \mathrm{p}^n(x,y) = \pi(y) \;\; \forall\, x \in S.$$

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This is great, but what does this **aperiodicity** refer to?!.

Period of a state

Definition: Given a Markov Chain, the **period** of a state x is the largest common divisor of the numbers in the set

$$I_x := \{n \in \mathbb{N} \mid \mathrm{p}^n(x,x) > 0\}$$

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Example: Remember the Ehrenfest chain? For any state in this chain $\mathrm{p}^n(x,x) > 0$ **if and only if** n is even. Thus, the period of every state is 2.

Example: If x is such that $\mathrm{p}^2(x,x) > 0$ and $\mathrm{p}^3(x,x) > 0$ then the period of x is 1.

Period of a state

Properties of the period

- The set I_x is closed under sums, that is, if $n \in I_x$ and $m \in I_x$, then $n + m \in I_x$.

Period of a state

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- The set I_x is closed under sums, that is, if $n \in I_x$ and $m \in I_x$, then $n + m \in I_x$.

- If $\mathrm{p}(x, x) > 0$, then x has period equal to 1.

Period of a state

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- If $\mathrm{p}(x, x) > 0$, then x has period equal to 1.

- If $x \mapsto y$ **and** $y \mapsto x$ then x and y have the same period. In particular, in an irreducible chain, all states have the same period.

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- If x has period 1, then there is a number n_0 such that $\mathrm{p}^n(x,x) > 0$ **for every number** $n \geq n_0$.

Period of a state

Properties of the period

Definition: If every state in a chain has period equal to 1, the chain is said to be **aperiodic**.

• If a chain with N states is irreducible and aperiodic, then $\mathrm{p}^n(x, y) > 0$ **for all** $n \geq N$ **and any states** x **and** y**.**