Before anything else, a little story

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Here is Newton's Universal Law of Gravitation

$$F = \frac{Gm_1m_2}{r^2}$$

(point masses attract one another with a force proportional to the product of their respective two masses and the inverse square of the distance between them)

Before anything else, a little story

Here is Newton's Universal Law of Gravitation

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Now, seriously, how did Newton even come up with this?!

Newton's Universal Law of Gravitation

$$F = \frac{Gm_1m_2}{r^2}$$

The discovery of this law represents a crucial point in the history of humanity. For instance,

- It predicted with unprecedented accuracy the location of planets in the sky, as well as the passing of comets.
- It led to a new era of astronomy and science in general.
- It was an important precedent to Albert Einstein's Theory of General Relativity, which showed us how the geometry of space and time is tied to gravity and matter.
- The predictions made by the theory were accurate enough to guide the navigation of the Apollo missions, hundreds of years after its discovery. The equation is widely used **everyday**.

Now, seriously, how did Newton even come up with this?!

$$F = \frac{Gm_1m_2}{r^2}$$

The apocryphal tale of Newton's discovery of the universal law of gravitation involved the falling of an apple.

What actually happened was a hard earned discovery which combined experimental observations by astronomers like Kepler and Brahe, and mathematics, including analytic and Greek geometry, and the newly developed differential calculus.

What was known at the time of Newton

- The Euclidean geometry of the Ancient Greeks, and importantly for us, the knowledge of conic sections.
- Descartes introduction of coordinates into geometry (a relatively recent event in Newton's days).
- Kepler's description of three empirical laws describing the behavior of the planets –based on astronomical data.

What was known at the time of Newton

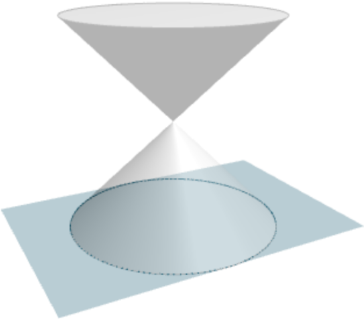
Apollonius' Conic Sections

*Apollonius of Perga (presently Bergama, Turkey) was a Greek geometer and astronomer who lived in the late 3rd/early 2nd century BC. Perhaps he is best known in our time for his work on **conic sections**.*

What was known at the time of Newton

Apollonius' Conic Sections

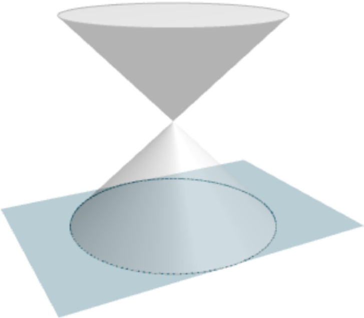
First: consider a double **cone**, just as the one below...



What was known at the time of Newton

Apollonius' Conic Sections

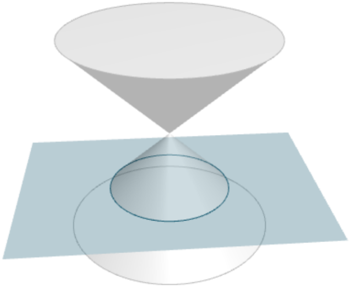
... as well as a **plane** that we are free to move around.



What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

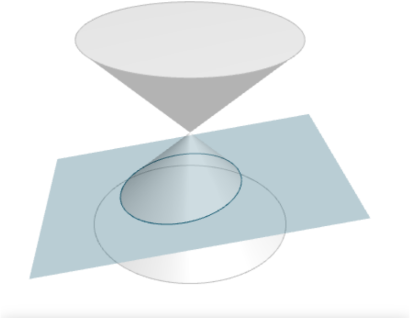


In this case, the curve turns out to be a **circle**.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

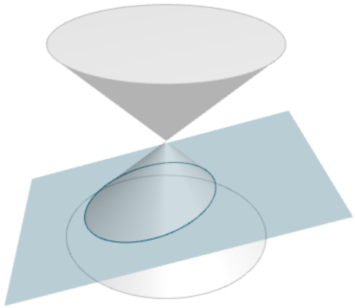


If we start tilting the plane, the circle turns into an **ellipse**.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

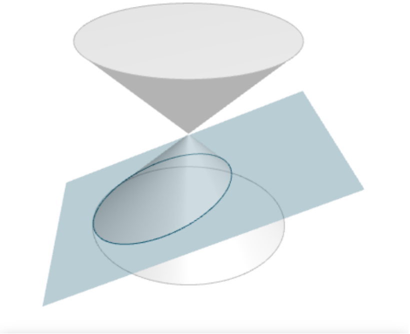


The more we tilt the plane, the more **eccentric** the ellipse.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

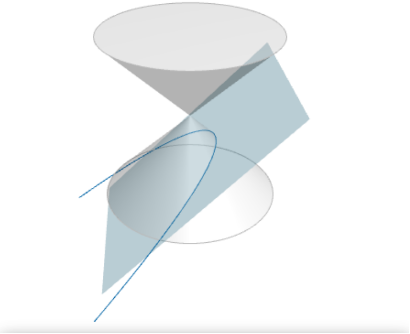


The more we tilt the plane, the more **eccentric** the ellipse.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

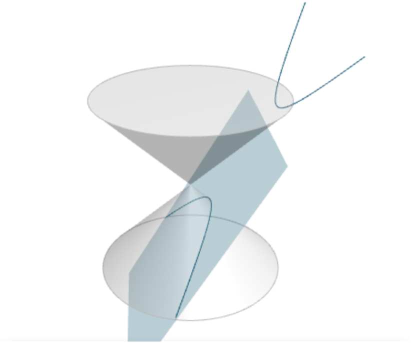


When the tilting is just right, the ellipse turns into a **parabola**!.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

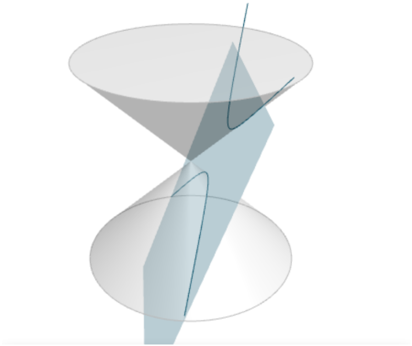


The slightest furhter tilt turns the parabola into a **hyperbola**.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.



Further tilting gives us a more eccentric hyperbola.

What was known at the time of Newton

Apollonius' Conic Sections

So, a conic section is the planar curve formed by the intersection of a plane with a double cone. The possible shapes are ellipses, parabolas, hyperbolas, and a pair of straight lines crossing each other.

What was known at the time of Newton

Apollonius' Conic Sections

Conic sections were widely studied by the ancient geometers, they were themselves object of study but also tools to solve various geometrical problems that were posed at the time.

More than a thousand years later, in the Renaissance, the study of the ancient greek books became a must for anybody interesting in what was known then as natural science (which is a mix of mathematics and physics).

What was known at the time of Newton

Apollonius' Conic Sections

Scientists (or "natural philosophers" as they called themselves) were guided by an ideal that the laws governing the universe must be expressable in terms of mathematics, and in terms of simple rules and beautiful geometric constructions.

This was true in particular, of scientists like the German astronomer Johannes Kepler, who was part of the scientific revolution of the 17th century.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

First Law: The motion of a planet describes an ellipse, with the sun located at one of its two focal points.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

Second Law: The trajectory along the orbit is such that, if one draws a line joining the planet to the sun, and keeps track of the region swept by it over time, then the area of this region is the same for all time intervals of equal length.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

Third Law: The orbit time T for the planet is such that its square is proportional to the cube of the major semiaxis of the ellipse.

What was known at the time of Newton

Hooke's harmonic motion

An early attempt to describe the movements of planets was done by Hooke…

it consisted of (what we know call) the second order differential equation

$$\ddot{f} = -k^2 f$$

Whose solutions are given by

$$f(t) = f(0)\cos(kt) + \frac{\dot{f}(0)}{k}\sin(kt)$$

What was known at the time of Newton

Description of an ellipse

Let us put the **First Law** in analytical form. Let us write

$$\begin{aligned} a &= \text{ semi-major axis} \\ b &= \text{ semi-minor axis} \\ e &= \text{ eccentricity} = \sqrt{1-(b/a)^2} \end{aligned}$$

Then, in polar coordinates, we have

$$r(\theta(t)) = \frac{a(1-e^2)}{1 - e\cos(\theta(t))}$$

where $\theta(t)$ is the angle given by the planet's position at time t.

What was known at the time of Newton

Description of an ellipse

What about $\theta(t)$? Here we use the **Second Law**. Note that

$$\mathrm{Area}(\theta_1, \theta_2) = \frac{1}{2} \int_{\theta_1}^{\theta_2} r(\theta)^2 \, d\theta$$

Then, by the **Second Law**

$$\mathrm{Area}(\theta(t), \theta(t+s)) \text{ is independent of } t.$$

Dividing by s and letting $s \to 0$, one concludes that

$$r^2(\theta(t))\dot{\theta}(t) \text{ is independent of } t$$

What was known at the time of Newton

Description of an ellipse

We now go back to Cartesian coordinates

$$x(t) = r(\theta(t))\cos(\theta), \ \ y(t) = r(\theta(t)\sin(\theta(t))$$

From the chain rule, and the relation $\dot{\theta}(t)r(\theta(t))^2 = c_0$, one gets

$$\dot{x}(t) = -c_0\sin(\theta(t)), \ \ \dot{y}(t) = c_0(\cos(\theta(t)) - e)$$

What was known at the time of Newton

Description of an ellipse

Taking derivatives again, and using the relation between $\dot{\theta}$ and $r(\theta)$ again, one obtains

$$\ddot{x} = -c\frac{1}{r(\theta)^2}\cos(\theta),$$
$$\ddot{y} = -c\frac{1}{r(\theta)^2}\sin(\theta).$$

for some constant c. Using the **Third Law**, one sees further c is independent of e and a.

What was known at the time of Newton

One more thing: circular orbits

First Law (for a circle)

$$x(t) = R\cos(\theta(t)), \;\; y(t) = R\sin(\theta(t)).$$

Second Law

$$\theta(t) = \omega t + \theta_0.$$

Third Law

$$\omega = cR^{-3/2}$$

The inverse square law follows. This was known before Newton!

What's the story?

1. A **problem** arising from the **physical** world: what, if any, are the mechanisms determining planetary motion?.
2. Empirical **observations**: Kepler's three laws.
3. **Attempts** at a theory: Copernicus' circles model, Hooke's law for harmonic motion.
4. A new **mathematical tool** is developed (Calculus), allowing Newton to find the Universal Law of Gravitation, which gives a full explanation for Kepler's laws.
5. The new theory not only agrees with available data, it starts making **predictions** considered astonishing at the time: the prediction by Halley of the arrival of a comet.

What's the story?

In the immortal words of Matt Damon:

How you like them apples?!

Welcome to
Math 456: Mathematical Modeling

Instructor: Nestor Guillen

What this class is about

Goal: To gain practice in the craft and science of using math models to describe, understand, and predict things.

What this class is about

Goal: To gain practice in the craft and science of using math models to describe, understand, and predict things.

Difficulty: This is really hard to do.

What this class is about

A **Mathematical Model** can refer to almost anything, e.g.

1. A function or a formula of some kind, involving various parameters and providing a numerical value as output.
2. A differential equation or dynamical system.
3. A probability distribution.
4. A stochastic process (randomness!, uncertainty in predictions and data)
5. Any other math structure (a Hilbert space with a self-adjoint operator).

What this class is about

Basically, a math model is any type of mathematics you can use to analyze and predict phenomena.

Challenges for this class

1. There are more types of math models than branches of math. You can't learn all of math first and then do some modeling!.
2. A specific model or family of models may come with parameters (e.g. the mass of the sun, mass of the earth) that need to be **inferred** empirically.
3. Reality is deeply complicated: depending on how much detail or accuracy we want, the underlying mathematical model can be very hard to analyze. For many important real life phenomena, the respective mathematical challenges are far beyond what mathematicians (even with the aid of computers) can do, e.g. turbelence.
4. There is a common pitfall of teaching through oversimplified toy models ("consider a spherical cow").

This semester

- We will study *some* ideas relevant to deterministic systems.
- We will study *some* ideas relevant to stochastic systems.
- We will get some practice in modeling, working in groups that will each focus on some preset modeling project (with presentations and written report later in the semester). The project will be some combination of

This semester

Part I. Theory (about 10 weeks)

1. Notion of a dynamical systems, generalities of modeling.
2. Examples of least action and other variational ideas in modeling.
3. A rapid review of probability.
4. Markov Chains: conditioning, the Markov property, and the Chapman-Kolmogorov equation
5. Examples of finite-state Markov chains
6. Stationary distributions
7. Long time behavior of chains and the ergodic theorem.
8. The Monte-Carlo method

Part II. Group projects (about 4 weeks)

Class set up and evaluation

My office: LGRT 1342. **Email**: nguillen@math.umass.edu.
Office hours (tent): Tuesday 1-3 pm, or by appointment.
"Email Office hours": Fri 10-11.

Your TAs and office hours info:

Lingchen Bu: Mondays 10 am-12 pm at LGRT 1323E.
John Lee: Wednesdays 10 am -12 pm at LGRT 1423O.

Class set up and evaluation

Evaluation:
Problem Sets 30%.
Midterm 30%. **Date:** April 5th (Thursday Week 10).
Project 40%.

The project will be done in groups of 3 and it will involve 3 important dates: submitting an abstract by the end of week 4, submitting an early report (with any code and references) by week 10, and a final report on week 15.

Problem Set Policies: Lowest homework grade is dropped. No late homeworks are accepted.

Class attendance is **mandatory**.

Class set up and evaluation

Class textbook:
Essentials of Stochastic Processes by Richard Durrett (2012 ed.)

Homeworks announced via email, due approx. every two weeks.

My **homepage**: for first day handout information, along with class slides, class plan, and upcoming homeworks:

http://people.math.umass.edu/ nguillen/

Back to some math

To finish this intro to the class, let me complement our discussion about Newton's laws with a preview of two important **random** models.

The Ising Model

$$\sigma : \Lambda \mapsto \{-1, 1\}$$

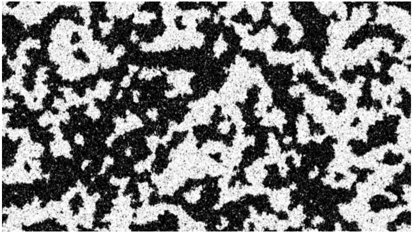
$$H(\sigma) = -\sum_{x \sim y} \sigma(x)\sigma(y)$$

The Ising Model

The smaller $H(\sigma)$ the more likely the configuration σ is.

$$P(\sigma) \sim e^{-\beta H(\sigma)}$$

A sample configuration:



The Ising Model

Beyond its origins in statistical mechanics:

1. All over physics (see: spin glasses)
2. Neurology (models for neural networks)
3. Machine learning (something something artificial intelligence something something complete)
4. Large networks (e.g. the internet with its pages and links)
5. Material science and engineering (multi-scale physics)

The Random Walk

The position after n steps is denoted by X_n, which is an integer.

$$X_0 = 0$$

Given X_n, $X_{n+1} = X_n + 1$ or $X_{n+1} = X_n - 1$ each with probability 1/2.
For instance, X_2 can take any of the values

$$-2, \;\; 0, \;\; 2$$

can take any of the values

$$-3, \;\; -1, \;\; 1, \;\; 3$$

Random walks in large networks

What if we have instead of the above "straight line" network, something more complicated?

For instance, what if I were to navigate wikipedia at random, what is the probability that I will go on clicking links at random 100 times without arriving once at the wikipedia page for **Philosophy**?

Associated to this is the idea of ergodicity, and mixing. The above question by the way, is somewhat related to google's search algorithm.

Math 456
Dynamical systems: examples and some generalities

A few dynamical systems and where they are used

The harmonic oscillator (pendulums, springs, circuits)

The nonlinear pendulum (more accurate model for a pendulum)

The N-body problem (multiple planet dynamics)

The exponential growth/decay equation (population growth, radioactive decay, compound interest)

The logistic equation (population growth)

Euler, Lagrange, and Kovalevskaya tops (spinning tops!)

The Lorenz attractor (metereology)

The Lotka-Volterra equation (predator-prey models in biology)

Description of the states of a system

In each of the examples above, one describes the state of the system at time t, via a vector made out of d-coordinates

$$x(t) = (x_1(t), \ldots, x_d(t))$$

The number of coordinates needed to describe the state of a system in a given instant is called the **dimension of the system.**

The description must be such that, roughly speaking, one has included all the relevant information about the system necessary to determine its instantaneous evolution.

Description of the states of a system

The exponential growth/decay equation: the current amount of the quantity considered $x(t)$.

$$\dot{x} = \lambda x$$

Description of the states of a system

The logistic equation: the size of the population $p(t)$.

$$\dot{p}(t) = rp(t)(1 - \frac{p(t)}{M})$$

Description of the states of a system

The harmonic oscillator: the displacement $x(t)$, and displacement velocity $v(t)$.

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = -\kappa^2 x(t)$$

Description of the states of a system

Nonlinear pendulum: angle $\theta(t)$ and angular velocity $\omega(t)$.

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\kappa^2 \sin(\theta)\end{aligned}$$

Description of the states of a system

How many coordinates do you need to describe the state of a gravitational system involving the earth, the moon, and the sun?

Description of the states of a system

How many coordinates do you need to describe the state of a gravitational system involving the earth, the moon, and the sun?

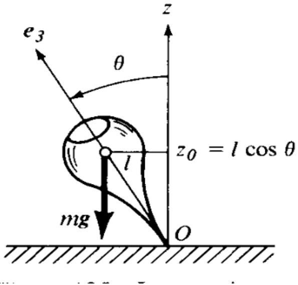
Answer: 6 coordinates for the sun, 6 coordinates for the earth, 6 for the sun. So, 18 coordinates.

What does the Newton's law of gravitation say for such a system?

Description of the states of a system

How many coordinates do you need to describe the state of a spinning top?

Hint: It depends on the symmetries of the top!



Description of the states of a system

Let us go back to a point made a few slides ago:

The description must be such that, roughly speaking, one has included all the relevant information about the system necessary to determine its instantaneous evolution.

Concretely, what the above means is that one can write

$$\dot{x}(t) = \mathbf{v}(x(t))$$

so, the rate of change of the coordinates of the system can be determined from the current state via a function $\mathbf{v}$, which depends on d-variables and returns as its value a d-dimensional vector.

Analysis of solutions

The chief purpose of our dynamical system is to make predictions, if we know the state of the system now, say, x_0, to determine $x(t)$ for some later time t, or to at least be able to say **something** about $x(t)$ for a later time.

Let us recall how this is done sometimes, as discussed in any basic differential equations class.

Analysis of solutions

Example: Let's consider a typical one dimensional situation solvable by separation of variables, we look for a function $y(t)$ (real valued) solving

$$\dot{y}(t) = g(y(t))$$

The idea is that, because of the chain rule, it makes sense to look for the antiderivative of

$$\frac{1}{g(y)}$$

as a function of y.

Analysis of solutions

Example: If

$$\frac{dG}{dy} = \frac{1}{g(y)}$$

Then the solution to the equation, $y(t)$, solves

$$\frac{d}{dt}G(y(t)) = \frac{\dot{y}(t)}{g(y(t))} = 1$$

Analysis of solutions

Example: therefore

$$G(y(t)) - G(y(0)) = t$$

Then, one hopes that the above function G can be easily inverted to arrive at a formula for $y(t)$ in terms of t and $y(0)$.

$$y(t) = G^{-1}(t - G(y(0)))$$

Analysis of solutions

Example: Let us see this in the most basic case, that of the linear equation

$$\dot{y} = ay$$

where a is some given number. Following the steps above yields

$$y(t) = e^{ta}y(0).$$

Analysis of solutions

Example: A more interesting example is the non-linear equation,

$$\dot{y} = y^2$$

What happens in this case? Well, we have

$$\frac{\dot{y}}{y^2} = 1 \Rightarrow \frac{d}{dt}\left(-\frac{1}{y(t)}\right) = 1$$

From where one gets the formula $y(t) = y(0)(1 - ty(0))^{-1}$.

Analysis of solutions

Except for pathological instances of systems with a poorly behaved $\mathbf{v}(x)$, a system will be "well posed" in the sense that one can find a function depending on time t, and the initial condition x_0, such that any solution to the system $\dot{x} = \mathbf{v}(x(t))$ is given by

$$x(t) = U(t, x_0), \;\; x_0 = x(0).$$

This function $U(t, x_0)$ is called by several names: **the evolution operator**, the **one-parameter flow** generated by $\mathbf{v}$, or **the one-parameter semigroup** for the system. As we saw earlier,

$$\dot{x} = ax \Rightarrow U(t, x_0) = e^{ta} x_0$$

Analysis of solutions

Of course, if your system is of dimension d, then $U(t, x_0)$ is a d-dimensional vector valued function.

For instance, what about the harmonic oscillator $\ddot{x} = -\kappa^2 x$?

Well, in this case

$$U(t, x_0, \dot{x}_0)$$
$$= (\cos(\kappa t)x_0 + \kappa^{-1}\sin(\kappa t)\dot{x}_0, -\kappa\sin(\kappa t)x_0 + \cos(\kappa t)\dot{x}_0)$$

Analysis of solutions

Then suppose we want to find a formula

$$x(t) = U(t, x_0)$$

for the non-linear pendulum

$$\ddot{x} = -\kappa^2 \sin(x).$$

In this case, is there a reasonable expression for $U(t, x_0)$?
…Not really!!

Analysis of solutions

The spinning top?

Analysis of solutions

The spinning top? Only some instances and after a lot of work.

The N-body problem?

Analysis of solutions

The spinning top? Only some instances and after a lot of work.

The N-body problem? **Even worse.**

In those cases, and if we want to analyze what happens near a special state of the system, say an equilibrium point, then one could do a **linear analysis**. Approximating the original equation with a linear one.

A few remarks about linear systems

As we know, the general linear system can be expressed as

$$\dot{x} = Ax$$

for a $d \times d$ matrix A. In this instance, the solution is given by

$$x(t) = e^{tA}x(0)$$

or, in other words, the evolution operator is the matrix

$$U(t) = e^{tA}$$

A few remarks about linear systems

Allow me to explain briefly the concept of the exponential of a matrix.

In one line, it simply means, for a given $d \times d$ matrix A,

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

A few facts about linear systems

Example: Suppose A is a matrix and x_0 a vector such that

$$Ax_0 = \lambda x_0$$

for some $\lambda \in \mathbb{R}$. Then,

$$e^A x_0 = e^\lambda x_0.$$

Example: What if A is a diagonalizable matrix? What can we say about e^A?

A few facts about (non)linear systems

Gronwall's lemma

Example:
Remember separation of variables? It allows us to write an explicit solution for

$$\dot{x} = a(x - x_0)$$

Example:
Consider what can be done if instead we had a function $x(t)$ which **does not solve** a differential equation, but for every time we have

$$\dot{x} \leq a(x - x_*)$$

Since the above is not an equation but an equality, one says that $x(t)$ satisfies a **differential inequality.**

A few facts about (non)linear systems

Gronwall's lemma

Example:
Assuming for a moment that $x > x_*$, we can rewrite the inequality as

$$\frac{d}{dt}(\log(x - x_*)) \leq a.$$

But then, integrating from 0 to t,

$$\log(x(t) - x_*) - \log(x_0 - x_*) \leq at$$

This can be rewritten as

$$x(t) - x_* \leq e^{at}(x_0 - x_*)$$

A few facts about (non)linear systems

Lyapunov functions and exponential relaxation

Example:
Take a d-dimensional system

$$\dot{x} = \mathbf{v}(x)$$

We say this system is a **gradient flow**, if there is some scalar function E such that

$$\mathbf{v} = -\nabla E$$

A few facts about (non)linear systems

Lyapunov functions and exponential relaxation

Example:

Consider, for given numbers $0 < \lambda_1 < \lambda_2$

$E(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$

$$\dot{x}_1 = -\partial_{x_1} E$$
$$\dot{x}_2 = -\partial_{x_2} E$$

A few facts about (non)linear systems

Lyapunov functions and exponential relaxation

Example:

$$
\begin{aligned}
y(t) &= x_1(t)^2 + x_2(t)^2 \\
\dot{y}(t) &= -2(x_1 \partial_{x_1} E + x_2 \partial_{x_2} E)
\end{aligned}
$$

Therefore,

$$\dot{y}(t) \leq -2\lambda_1 y(t)$$

A few facts about (non)linear systems

Lyapunov functions and exponential relaxation

Example:
Gronwall's Lemma then tells us that solutions for this system decay at least exponentially fast towards the origin, that is

$$|x(t)| \leq e^{-\lambda_1 t}|x(0)|$$

(here we are using the notation $|x(t)| = \sqrt{x_1(t)^2 + x_2(t)^2}$)

What we discussed today

1. Explicit formulas that represent a solution are useful –when you can find them.
2. Systems admitting an explicit formula are the exception and not the rule.
3. Linear systems are a case where there are explicit formulas.
4. The higher the dimension of the problem, and the more nonlinear it is, the harder it is to analyze.
5. At the same time, if a phenomenon is more complex, then it requires a higher dimensional as well as non-linear description.
6. Often, to a first approximation, a nonlinear model can be well approximated by a linear one.

What we discussed today

7. For a linear system, the eigenvalues of the underlying matrix greatly determine the properties of the system. Moreover, for a non-linear system the stability of an stationary point can often be determined by an approximating linear system.
8. Nonlinear systems sometimes have some additional feature that allows you to, if not to write down a formula for the solution, at least understand **some things** about their behavior.
9. Broaden your mind: search not just for equalities, but also inequalities. Inequalities don't tell you everything, but for many questions they are as useful as a formula.