# Math 456 Spring 2017

## (Practice) Midterm

You need to do **FOUR** of the problems below, each problem is worth 25 points. You will not get credit more for work in more than 4 problems, but your grade will be determined by the best 4 answers, so you are encouraged to answer as many question as possible. The exam lasts 1 hour and 10 minutes.

(1) Consider a real $3 \times 3$ matrix $A$.
1) Assume that all of the eigenvalues of $A$ are real and strictly negative. Compute the following limit

$$\lim_{t \to \infty} e^{tA} v,$$

where $v = (1, 0, 0)$.
2) Prove or disprove by a counter example: if one of the eigenvalues of $A$ has non-zero imaginary part, then that always means there is a vector $v$ such that $e^{tA}v$ does not converge to zero as $t \to \infty$.

(2) Consider $x(t)$ the solution of $\dot{x} = Ax + b$, where $x(0) = (0, 0, 0, 0)$ and

$$A = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

in two different cases (i) when $b = (1, 0, 0, 0)$ and (ii) when $b = (0, 0, 1, 0)$.
For each case, compute the solution $x(t)$, and explain what happens as $t \to \infty$ (namely: is there a limit at all? and if there is one, what is it?).

(3) Consider a chain $X_1, X_2, \ldots$ with state space $S = \{1, 2, 3, 4\}$ pictured as points on the line. The probability of jumping left or right is always equal to $1/2$, save for *periodic* conditions at the boundary points: if one is at state $x = 1$ and jumps left, one ends at state $x = 4$, and if one is at state $x = 4$ and jumps right, then one ends at state $x = 1$.

Let $Y_n$ be the Markov chain obtained by defining $Y_n := X_{2n}$. Explain why $Y_n$ is itself a Markov chain, and write down its transition probability matrix.

(4) Provide an example of each of the following
1) A $3 \times 3$ transition probability matrix corresponding to an irreducible chain.
2) A $3 \times 3$ transition probability matrix which has exactly one recurrent state.
3) A $4 \times 4$ transition probability matrix which is both aperiodic and irreducible.

(5) Consider the Ehrenfest chain with $N = 4$, whose transition matrix is given by

$$\text{p} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Compute the probabilities of the following paths

| | $X_0$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|---|
| Path 1 | 0 | 1 | 2 | 3 |
| Path 2 | 1 | 4 | 1 | 4 |
| Path 3 | 2 | 3 | 2 | 3 |
| Path 4 | 2 | 3 | 2 | 1 |

(6) Consider the chain with state space $S = \{1, 2, 3, 4\}$, and with transition matrix given by

$$\begin{pmatrix} 0.8 & 0.1 & 0 & 0.1 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.4 & 0 & 0.3 & 0.3 \end{pmatrix}$$

Which states are recurrent? Which states are transient?. Compute the period of each of the four states.

(7) Consider the chain with state space $S = \{1, 2, 3, 4\}$, and with transition matrix given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.4 & 0 & 0.3 & 0.3 \end{pmatrix}$$

Which states are recurrent? Which states are transient?. Find a stationary distribution for this chain.

(8) Take the Ehrenfest chain from Problem 5. Write down its stationary distribution. Then, say what is the approximate percentage of the time that the chain occupies the state corresponding to $N = 3$ as the number of time steps goes to infinity.

(9) Choose one of the transition matrices from either problem 6 or problem 7. Then, for every $x \in \{1, 2, 3, 4\}$, compute the limit of

$$\lim_{n\to\infty} \mathrm{p}^n(2, x)$$

(10) Consider the following two chains with state space $S = \{1, 2, 3, 4\}$,

$$a) \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix} \qquad b) \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}$$

For each chain, write down which states are transient and which are recurrent.

(11) Consider the Markov Chain over $S = \{1, 2, 3, 4, 5\}$

$$\begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0 & 0 & 0.8 & 0.1 & 0.1 \\ 0 & 0 & 0.3 & 0.4 & 0.3 \end{pmatrix}$$

Let $\Delta$ denote the Laplacian associated to this chain. Find the unique function $f : S \to \mathbb{R}$ with the following properties

$$\Delta f(x) = 0 \text{ if } x \in \{2,3,4\},\ f(1) = 0,\ f(5) = 1.$$

Then, provide a probabilistic interpretation for $f(2)$.

(12) Consider the chain with state space $\{1,2,3,4\}$ and transition matrix given by

$$\begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.2 & 0.5 & 0.3 & 0 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix}$$

List all the paths of length 3 going from state 1 to state 4 which have positive probability (e.g. $1,1,3,4$ is one such path while $1,2,3,4$ is not), use this to compute $\mathrm{p}^3(1,4)$.