**Math 456 Spring 2018. Problem Set 4.**

(1) Consider a general chain with state space $S = \{1, 2\}$, and transition matrix

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

where we assume that $0 < a < 1, 0 < b < 1$.

(a) Verify by direct calculation that $\pi = (b/(a+b), a/(a+b))$ is a stationary distribution.

(b) Use the Markov property to show that

$$\mathrm{P}(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\left(\mathrm{P}(X_n = 1) - \frac{b}{a+b}\right).$$

(c) Use the recursive relation in b) to show that

$$\lim_{n\mapsto\infty} \mathrm{P}(X_n = 1) = \frac{b}{a+b}.$$

In fact, use b) to show that the convergence to the limit happens **exponentially fast**, i.e. find $\lambda > 0$ and $C > 0$ such that for all $n \in \mathbb{N}$,

$$\left|\mathrm{P}(X_n = 1) - \frac{b}{a+b}\right| \le Ce^{-\lambda n}.$$

(2) To make a crude model for a patch of forest we might introduce states $0 =$ grass, $1 =$ bushes, $2 =$ small trees, $3 =$ large trees, and write down a transition matrix like the following:

$$\mathrm{p} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/24 & 7/8 & 1/12 & 0 \\ 1/36 & 0 & 8/9 & 1/12 \\ 1/8 & 0 & 0 & 7/8 \end{pmatrix}$$

Then,

(a) Prove that this distribution is irreducible

(b) Find the stationary distribution.

*Note: The idea behind this matrix is that if left undisturbed a grassy area will see bushes grow, then small trees, which of course grow into large trees. However, disturbances such as tree falls or fires can reset the system to state* 0.

(3) A bank classifies loans as paid in full (F), in good standing(G), in arrears(A), or as a bad debt (B). Loans move between the categories according to the following transition probability:

| | **F** | **G** | **A** | **B** |
|---|---|---|---|---|
| **F** | 1 | 0 | 0 | 0 |
| **G** | 0.1 | 0.8 | 0.1 | 0 |
| **A** | 0.1 | 0.4 | 0.4 | 0.1 |
| **B** | 0 | 0 | 0 | 1 |

What fraction of loans in good standing are eventually paid in full? What is the answer for those in arrears?.

(4) Consider the numbers $1, 2, \ldots, 12$ written around a ring as they usually are on a clock. Consider a Markov chain that at any point jumps with equal probability to the two adjacent numbers. (a) What is the expected number of steps that $X_n$ will take to return to its starting position? (b) What is the probability $X_n$ will visit all the other states before returning to its starting position?