Math 456 Spring 2018. Problem Set 2.

(1) A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

(2) Suppose that the probability it rains today is 0.3 if neither of the last 2 days was rainy, but 0.6 if at least one of the last 2 days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last 2 days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday?.

(3) We repeatedly roll two symmetric four-sided dice with numbers $\{1, 2, 3, 4\}$ on them. Let Y_k be the sum on the k-th roll, and $S_n = Y_1 + \ldots + Y_n$ be the total of the first n rolls. Let X_n, taking values in $\{0, 1, 2, 3, 4, 5\}$ be defined as S_n *modulo* 6 (i.e. the residue left when dividing X_n by 6). **Write down** the transitiion probability matrix for X_n.

(4) (BONUS PROBLEM: Prob. #4 from Set 1 revisited)
   (a) Compute the exponential e^{tA} where A is as in Prob. #4 Set 1, without having to compute eigenvectors. Do this by using the power-series formula for the exponential, and the fact that the powers of A have the form

$$A^n = \begin{pmatrix} 5^n & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n \end{pmatrix}.$$

   (b) Suppose you have a solution $x(t)$ to the (scala) equation $\dot{x} = ax + b$, where a and b are real constants. Find a real valued function $u(t)$ such that

$$\frac{d}{dt}(ux) = u(t)b \qquad (1)$$

   Hint: Suppose you already have that u, apply the chain rule and see what equation u needs to solve.

   (c) Integrate the identity (1) from time 0 to some arbitrary time t. From here, conclude that $x(t)$ can be written as

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-s)} b \, ds$$

   (d) Now suppose that $x(t)$ is a vector valued function solving the system $\dot{x} = Ax + b$, where A is a matrix and b is some constant vector. Find a (real) matrix-valued function $U(t)$ such that

$$\frac{d}{dt}\left(U(t)x(t)\right) = U(t)b \qquad (2)$$

   (e) Proceed as in step (2) and show that the vector $x(t)$ can be represented as

$$x(t) = e^{At}x(0) + \int_0^t e^{(t-s)A} b \, ds$$