Math 456 Spring 2018. Problem Set 1.

(1) Let A be a 3×3 diagonal matrix whose diagonal elements are denoted $\lambda_1, \lambda_2, \lambda_3$. For $t \in \mathbb{R}$ let $U(t)$ denote

$$U(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$$

By direct computation (just straight matrix multiplication and differentiation), check that

$$\dot{U} = AU = UA.$$

(2) Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Prove that $U(t) = e^{tA}$ is given by

$$U(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

Hint: Note that $A^2 = I$, and use this to simplify the series that gives rise to e^{tA}. Also, review the power series corresponding to $\cos(t)$ and $\sin(t)$.

(3) Let $U(t)$ denote a $N \times N$-matrix valued function for $t \in \mathbb{R}$, and suppose that each of its components is a differentiable function. Let $x(t)$ denote a differentiable function of $t \in \mathbb{R}$ taking values in $\mathbb{R}^N$. Then, check that if $y(t) = U(t)x(t)$ then

$$\dot{y}(t) = \dot{U}(t)x(t) + U(t)\dot{x}(t).$$

Hint: This is nothing but Leibniz rule.

(4) Consider the differential equation $\dot{x} = Ax + b$, where

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}.$$

and use variation of parameters to find a explicit formula for the solution starting from $x_0 = 0$.

(5) Match each function on the left to the differential equation it solves on the right (showing your computations in each case, of course)

a) $x(t) = x_0/(1 - x_0 t)$	1) $\ddot{x} = 4x$
b) $x(t) = \frac{1}{2}x_0(e^{2t} + e^{-2t})$	2) $\dot{x} = x(1 - x)$
c) $x(t) = x_0 \cos(2t)$	3) $\dot{x} = x^2$
d) $x(t) = e^t x_0/(1 + (e^t - 1)x_0)$	4) $\ddot{x} = -4x$.

Hint: This is simply an exercise of computing derivatives and plugging the derivatives in each equation. Each function on the left solves exactly one equation on the right, try to find those which solve the simplest (linear) equations on the right first.

(6) A positive function $x(t)$ with $x(0) = 1$ defined for $t \in (0, 1)$ is such that

$$\dot{x}(t) \geq x(t)^2 \ \ \forall\, t \in (0, 1).$$

Show that $x(t) \to \infty$ as $t \to 1$. *Hint: Divide both sides of the inequality by $x(t)^2$, and integrate with respect to t.*