Texas State University
MATH 3323: Differential Equations
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Problem Set 6

NOTE: This problem set is due Wednesday, October 16th.

(1) For each second order equation given, write it as an equivalent first order system

(a) $\ddot{x} - 2t\dot{x} + 7x = \sin(t)$,
(b) $\ddot{x} - 2\dot{x} + 2x = 0$,
(c) $\ddot{x} + \dfrac{9}{x^2} = 0$,
(d) $\ddot{x} + y = 0,\ \ddot{y} + \sin(x) = 1$.

(2) For each Initial Value Problem below, write down the third Picard iteration $x_3(t)$ obtained where $x_0(t)$ is taken to be the given initial condition.

(a) $\dot{x} = 2x + 1,\ x(0) = 3$,
(b) $\dot{x} = -x^2,\ x(0) = 1$,
(c) $\dot{x} = x - x^2,\ x(0) = 2$,
(d) $\dot{x} = \sin(x),\ x(0) = 0$.

(3) Find the solution to $\dot{x} = Ax$ with initial value $x(0) = x_0$ for each A and x_0 given below

(a) $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 7 & 3 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,

(b) $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 3 & 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$,

(c) $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & -1 & -4 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

(4) For each linear second order system below, write it as first order system and use that representation to find the general solution

(a) $\ddot{x} + 4x = 0$,
(c) $4\ddot{x} - 4\dot{x} + -3x = 0$,
(d) $4\ddot{x} + 17\dot{x} + 4x = 0$.

(5) (BONUS) Consider the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Let P denote the matrix $P = A - 3\mathrm{I}$ (I denotes the identity matrix). Then

(a) Show by direct computation that P^2 is the zero matrix – i.e. all its entries are zero.
(b) Compute a formula for e^{tP} (since $(tP)^2 = 0$, the series simplifies considerably!)

(c) Use this to compute a simple formula for e^{tA}.

(6) (BONUS) Consider a two dimensional nonlinear system given by

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -H'(x)\end{aligned}$$

If $(x(t), y(t))$ is any solution to this, show that there is some constant c such that

$$\frac{1}{2}y(t)^2 + H(x(t)) = c \text{ for all } t.$$

Apply this to the following problem: you are given a pendulum (take $m = 1$) which starts from a horizontal position (i.e. lying at an angle of $\pi/2$ away from equilibrium) and is initially at rest (i.e. the initial angular velocity is zero). The dynamics of the pendulum (θ = angle, ω = angular velocity) are governed by the equations

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\sin(\theta).\end{aligned}$$

What is the angular velocity at the moment when the pendulum is perfectly vertical?