Texas State University
MATH 3323: Differential Equations
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Problem Set 4

(1) Consider the real valued functions $x_1(t) = e^{-2t}$ and $x_2(t) = e^{-3t}$, then

(a) By direct computation, check that each function solves

$$\ddot{x} + 5\dot{x} + 6x = 0$$

(b) Find α_1 and α_2 such that the function given by

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

solves

$$\begin{aligned} &\ddot{x} + 5\dot{x} + 6x = 0 \\ &x(0) = 2,\ \dot{x}(0) = 9. \end{aligned}$$

(2) Check whether the following families of functions of t are linearly independent or not

(a) $t^2 + 1,\ 2t,\ 4(t+1)^2$

(b) $\sin(t)\cos(t),\ \sin(2t) + \cos(2t),\ \cos(2t)$

(c) $e^{2t},\ e^{-2t},\ 2e^{t}$

(d) $2e^{t},\ 3\cosh(t),\ 13\sinh(t)$

(e) $\dfrac{1}{t^2-1},\ \dfrac{1}{t+1},\ \dfrac{1}{t-1}$

(3) In each item below, compute the derivative of the given vector-valued function $\mathbf{x}$ (whose components are denoted $x_1(t)$ and $x_2(t)$) and match it to the differential equation it solves from the list on the right column

i) $\mathbf{x}(t) = \begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$

ii) $\mathbf{x}(t) = \begin{pmatrix} e^{2t}(t+1) \\ e^{2t} \end{pmatrix}$

iii) $\mathbf{x}(t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$

iv) $\mathbf{x}(t) = \begin{pmatrix} t + \frac{1}{2}(e^{2t} - 1) \\ \frac{1}{3}(e^{3t} - 1) \end{pmatrix}$

a) $\dot{x} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} x$

b) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) + \frac{\sqrt{2}}{2} \end{cases}$

c) $\dot{x} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} x + \begin{pmatrix} -2t \\ 1 \end{pmatrix}$

d) $\dot{x} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$

(4) Consider the matrix-valued functions of t

$$\mathbf{A}(t) = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & 2e^{-t} \\ 0 & 2e^{-t} & 2 \end{pmatrix} \quad \mathbf{B}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{t} \\ 0 & e^{-t} & -1 \end{pmatrix}$$

Then, compute the following expressions

a) $\mathbf{A} - 2\mathbf{B}$	**d)** $\int_0^1 \mathbf{B}(t)\, dt$
b) $\mathbf{AB}$	**e)** $\frac{d}{dt}(\mathbf{AB})$
c) $\frac{d}{dt}\mathbf{A}$	**f)** $(\frac{d}{dt}\mathbf{A})\mathbf{B} + \mathbf{A}(\frac{d}{dt}\mathbf{B})$

(5) (BONUS) Consider the function

$$x(t) = e^{2t}$$

Using the chain rule, show that given any numbers A, B, and C, then for all t we have the formula

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = (A4 + B2 + C)x$$

Determine similar formulas for $A\ddot{x}(t) + B\dot{x}(t) + Cx(t)$ for the following functions

$$e^t,\ e^{-t},\ e^{3t},\ e^{-3t}.$$

Based on your findings, try to find a (non-zero) function $x(t)$ solving the equation

$$\ddot{x} - 5\dot{x} - 24x = 0.$$