Texas State University
MATH 3323: Differential Equations
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Problem Set 2

(1) Write down a explicit formula for the solution $x(t)$ of each initial value problem

$$\text{a) } \dot{x} = (1-2t)x^2, \quad x(0) = -3$$

$$\text{b) } \dot{x} = \frac{tx}{\sqrt{1+t^2}}, \quad x(0) = 1$$

$$\text{c) } \dot{x} = 5t^{-1}x, \quad x(0) = 5$$

(2) Find the function $y(x)$ such that $y(0) = 0$ and which solves the equation

$$y'(x) = \frac{2-e^x}{3+2y(x)}$$

Once you find $y(x)$, find the value of x where it attains its maximum value.

(3) Consider the nonlinear differential equation

$$\dot{x} = \tfrac{1}{3}x(\tfrac{7}{23} - x)$$

Then

(a) Find the general formula for the solution (in terms of the initial value $x(0)$).
(b) When $x(0) = 1$, what happens with $x(t)$ as $t \to \infty$?
(c) Find a solution $x(t)$ of the equation which is constant in time.
(d) Give an example of initial value $x(0)$ so that $\lim_{t\to\infty} x(t) = 0$.

(4) (BONUS) Let $x_1(t)$ and $x_2(t)$ be two solutions to the equation

$$\dot{x} = f(x),$$

where all know about $f(x)$ is there is some $L > 0$ such that

$$f(x) - f(y) \leq L(y-x) \text{ whenever } x < y.$$

Show the following "differential inequality" holds

$$\frac{d}{dt}(x_1 - x_2)^2 \leq -2L(x_1 - x_2)^2.$$

Use this to show the following inequality for solutions

$$|x_1(t) - x_2(t)| \leq e^{-Lt}|x_1(0) - x_2(0)|, \quad \text{for } t > 0.$$

What do you think is the significance of this inequality? For the sake of concreteness, think for a second $x_1(t)$ and $x_2(t)$ represent the state of some physical system, what does this last inequality say about the behavior of the state of the system as time increases?

Hint: Once you obtain the differential inequality, use problem 5 from Problem Set 1.