Texas State University
MATH 3323: Differential Equations
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Problem Set 10

(1) Find the Laplace transform of each given function (do not forget to show your work, most of these calculations follow by a straightforward calculation, but you may use any of the identities on table 6.2.1 of the textbook or any of the identities given in class)

(a) $f(t) = te^{2t} - t^2e^{-4t}$
(b) $f(t) = 2t^2 + 2t + 4$
(c) $f(t) = e^{2t}\sin(4t) + \cos(t)$
(d) $f(t) = h_1(t) - h_2(t)$

Note: The function $h_\alpha(t)$ is the function which is equal to 0 as long as $t < \alpha$ and equal to 1 as long as $t \geq 1$, it is denoted by u_c in the book (c takes the place of α).

(2) Find the inverse Laplace transform of the following functions

(a) $F(s) = \dfrac{5!}{(s-4)^5}$
(b) $F(s) = \dfrac{1-2s}{s^2+4s+5}$
(c) $F(s) = \dfrac{e^{-s}+e^{-4s}}{s}$
(d) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

(3) Solve the following initial value problems

(a) $y'' + 4y = 1 - h_{3\pi}(t),\ \ y(0) = 1,\ y'(0) = 0$
(b) $y'' + y = \sin(t) - h_{2\pi}\sin(t-2\pi),\ y(0) = 0,\ y'(0) = 1$

(4) (BONUS) Find the inverse Laplace transform of the following functions

(a) $F(s) = \dfrac{4}{(s^2+1)(s^2+4)}$
(b) $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)}$

(5) (BONUS) Solve the following initial value problems

(a) $y'' - 4y = e^t + e^{-t},\ \ y(0) = 1,\ y'(0) = 0$
(b) $y'' + 100y = \sin(4t) + \cos(4t),\ y(0) = 0,\ y'(0) = 1$

(6) (BONUS) Solve the initial value problem

(a) $y'' + y' + \frac{5}{4}y = (1 - h_\pi(t))e^{-t}\sin(t),\ y(0) = 0,\ y'(0) = 0$

(b) $y'' + y' + \frac{5}{4}y = t - h_{\pi/2}(t)\left(t - \frac{\pi}{2}\right),\ y(0) = 0,\ y'(0) = 2$

(7) (BONUS) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic with period T if T is a number such that $f(t+T) = f(t)$ for all $t \in \mathbb{R}$. Show that if f is periodic with period T then

$$\mathcal{L}(f(t))(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Use this to compute the Laplace transform $\mathcal{L}(f)$ of the function f given by

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2, \end{cases} \quad \text{with } f(t+2) = f(t) \text{ for all } t.$$