Warm up: How many functions solve the equation

$$(*) \quad \dot{x} = x \, ?$$

It is easy to chek that any function of the form

$$x(t) = c e^{t}$$

(c being an arbitrary constant) solves this equation (so $2e^t$, $2019e^t$, $17e^t$ all solve it). Are there any others?

Suppose $x(t)$ is some function solving $(*)$, and that $x(t) \neq 0$ from all t, then we can rewrite $(*)$ as

$$\frac{\dot{x}}{x} = 1$$

The expression on the left is equal to:

$$\frac{d}{dt} \log(x(t)) \quad \text{(apply the chain rule!)}$$

so

$$\frac{\dot{x}}{x} = 1 \implies \frac{d}{dt}\log(x(t)) = 1$$

$\implies$ (integrating both sides of the equation)

$$\boxed{\log(x(t)) - \log(x(0)) = t} \quad (**)$$

So, from (*) (a differential equation) we have arrived at (**), which is an equation where no derivatives of $x(t)$ appear. Solving for $x(t)$ we see that

$$\log(x(t)) - \log(x(0)) = t$$

$$\implies \log\left(\frac{x(t)}{x(0)}\right) = t$$

$$\implies \frac{x(t)}{x(0)} = e^{t}$$

$$\implies x(t) = x(0)\, e^{t}.$$

In conclusion: as long as they don't vanish, all solutions of $\dot{x} = x$ are given by ce^t, c a constant.

What is a differential equation and what does it mean to solve one?

Types of Equations

$\boxed{1\text{-}d}$ First Order Differential Equation (1-d = 1 dimension, one unknown $x(t)$)

These take the form

$$\dot{x}(t) = V(x(t), t)$$

for some function V.

EX's (*) $\dot{x}(t) = x(t)^2$ ← this v does not depend on t!
(in this case $V(x,t) = x^2$)

(*) $\dot{x} = t \sin(x(t))$
(in this case $V(x,t) = t \sin(x)$)
note t dependence

$\boxed{1\text{-}d}$ Linear First Order Differential Equation

A subclass of equations of the previous class. These are first order differential equations where $Y(x,t)$ is a linear function of x: $\dot{x} = p(t)\,x + q(t)$

$\boxed{\text{EX's}}$

(*) $\dot{x}(t) = t\,x(t) + 1$

(*) $\dot{x}(t) = \frac{1}{1+t^2}\,x(t) + \frac{3}{1+t^2}$

$\boxed{2\text{-}d}$ Systems of ~~[illegible]~~ First Order Equations

EX's

(*) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$

(*) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) \end{cases}$

For these equations we have two unknowns, $x_1(t)$ and $x_2(t)$, and an equal number of equations.

What does it mean to solve a differential equation?

It can mean several things.

1. (Formulas) To find a explicit formula for a solution $x(t)$ in terms of some initial value(s) (such as $x(0)$ and $\dot{x}(0)$), that is, if such a formula can even be found.

2. (Existence and uniqueness) Are there solutions? Sometimes an equation doesn't have any solutions, other times they have solutions, but no "simple" or practical formulas exist for such solutions

3. (Qualitative Properties) Ok, if solutions exist, what can we say about them?, are they oscillating? ~~do~~ do they converge to some value as $t \to \infty$? are there interesting properties of solutions, such as the "law of conservation of energy" for some physical equations?

4. (Numerical Approximation) If there is a solution, can we write a numerical algorithm to approximate the values of solutions? (say, in a computer?)

In this course we will talk a bit about these 4 points, with an emphasis on the 1st one.

We will start by studying the best known methods for finding formulas for solutions of 1st order linear differential equations

1st Order Linear Equations

$$\dot{x} = p(t)\,x + q(t)$$

Two key methods:

1. Integrating Factor
2. Separation of Variables

We will study these methods over the next few classes.

Integrating Factor (Part 1)

$\boxed{\text{EX}}$ Let us try to find – as we did for $\dot{x} = x$ earlier – all functions $x(t)$ solving the differential equation

$$(1+t^2)\frac{dx}{dt} + 2t\,x = 1+t$$

What can be done? Let us try to, as we did with $\frac{\dot{x}}{x}=1$ to "recognize" a clear derivative on the left hand side of the equation

For this, we recall the product rule:

$$\frac{d}{dt}\left(f(t)g(t)\right) = f(t)\left(\frac{dg(t)}{dt}\right) + g(t)\left(\frac{df(t)}{t}\right).$$

Note that

$$\frac{d}{dt}\left(1+t^2\right) = 2t$$

that is, the derivative of the first "coefficient" on the equation $(1+t^2)$ is equal to the second coefficient $(2t)$.

On account of the product rule, this means that if $x(t)$ solves

$$(1+t^2)\frac{dx}{dt} + 2t\,x = 1+t$$

then it also solves

$$\frac{d}{dt}\left((1+t^2)x\right) = 1+t$$

and that's it! (compare to $\frac{\dot{x}}{x} = \frac{d}{dt}\log(x)$)

So we can integrate both sides of the equation* and see that

(*we integrate from 0 to t)

$$(1+t^2)x(t) - x(0) = t + \frac{t^2}{2}$$

solving for $x(t)$, we obtain the formula

$$\boxed{x(t) = \frac{x(0) + t + \frac{t^2}{2}}{1+t^2}}$$

which gives all solutions to the equation.

$\boxed{\text{EX}}$ The same trick as before can be applied to many other differential equations, such as

$$\left(\sin(x)x\right)^{\bullet} = \sin(t)\,\dot{x} + \cos(t)\,x = 1$$

or

$$\left(e^{2t}x\right)^{\bullet} = e^{2t}\,\dot{x} + 2e^{2t}x = e^{-2t}.$$

The "Integrating factor" method allows one to solve any equation of the form

$$\dot{x} = p(t)\,x + q(t)$$

and it consists in making the equation look like the ones in the examples.

<u>Note</u> (preliminary) The differential equation

$$\dot{x} = \lambda x \quad (\lambda \text{ a constant})$$

is solved by the function

$$x(t) = e^{\lambda t}.$$

We are going to explain the method when $p(t)$ is a constant, denoted by λ.

$$\boxed{\dot{x} = \lambda x + q(t)}$$

We first rewrite the equation as

$$\dot{x} - \lambda x = q(t)$$

We would like for the left hand side to resemble an expression coming from the product rule, so we multiply both sides by a (for now unknown) factor $\mu(t)$,

$$\mu \dot{x} - \lambda \mu x = \mu q \qquad (*)$$

if μ is chosen so that

$$\dot{\mu} = -\lambda \mu$$

then $(*)$ above is equivalent to the equation

$$\frac{d}{dt}(\mu x) = \mu q$$

from the note we know that

$$\mu(t) = e^{-\lambda t}$$

solves $\dot{\mu} = -\lambda\mu$, so we arrive at

$$\frac{d}{dt}\left(e^{-\lambda t} x\right) = e^{-\lambda \tau} q$$

Now we can integrate both sides, so, integrating from 0 to τ we have

$$e^{-\lambda t} x(t) - e^{0} x(0) = \int_0^t e^{-\lambda s} q(s)\, ds$$

Multiplying both sides by $e^{\lambda t}$ and moving $x(0)$ $(= e^{0} x(0))$ to the right side, we obtain the formula

$$\boxed{x(t) = e^{\lambda t} x(0) + e^{\lambda t} \int_0^t e^{-\lambda s} q(s)\, ds}$$

Let's see this method in a concrete example.

EX (see Problem Set 1, for comparison)

$$\dot{x} = -\frac{2}{3}x + e^{-2/3 t}$$

In this case $\lambda = -\frac{2}{3}$, so we write

$$\dot{x} + \frac{2}{3}x = e^{-2/3 t}$$

and multiply both sides by $e^{-\lambda t} = e^{\frac{2}{3}t}$,
and we get

$$e^{\frac{2}{3}t}\dot{x} + \frac{2}{3}e^{\frac{2}{3}t}x = \cancel{e^{\frac{2}{3}t}} \cdot \cancel{e^{-\frac{2}{3}t}} = 1$$

So we have

$$\frac{d}{dt}\left(e^{\frac{2}{3}t}x\right) = 1$$

Integrating from 0 to t (note: we are calling the integration variable "s")

$$\int_0^t \frac{d}{ds}\left(e^{\frac{2}{3}s}x(s)\right)ds = \int_0^t 1\, ds$$

$$e^{\frac{2}{3}t}x(t) - e^{\frac{2}{3}\cdot 0}x(0) = t - 0$$

$$\Rightarrow \quad e^{\frac{2}{3}t}x(t) = x(0) + t$$

$$\Rightarrow \quad x(t) = e^{-\frac{2}{3}t}x(0) + e^{-\frac{2}{3}t}t$$

Let us do another explicit example.

<u>EX</u>: Solve: $\dot{x}(t) = 7x(t) + \cos(t)$

In this case $\lambda = 7$, and rearranging the equation and multiplying both sides by $e^{-\lambda t}(= e^{-7t})$ we have

$$e^{-7t}\dot{x} - 7e^{-7t}x = e^{-7t}\cos(t)$$

This is the same as

$$\frac{d}{dt}\left(e^{-7t}x\right) = e^{-7t}\cos(t).$$

Integrating from 0 to s, this equation becomes

$$\int_0^t \frac{d}{ds}\left(e^{-7s}x(s)\right)ds = \int_0^t e^{-7s}\cos(s)\,ds$$

$$\|$$

$$e^{-7t}x(t) - x(0) = \int_0^t e^{-7s}\cos(s)\,ds$$

Using integration by parts, we look for the other integral; we have:

$$\int e^{-7s}\cos(s)\,ds = -\frac{1}{7}e^{-7s}\cos(s) + \frac{1}{7}\int e^{-7s}(-\sin(s))ds$$

and

$$\int e^{-7s}\sin(s)\,ds = -\frac{1}{7}e^{-7s}\sin(s) + \frac{1}{7}\int e^{-7s}\cos(s)\,ds$$

substituting this in the previous formula:

$$\int e^{-7s}\cos(s)ds = -\frac{1}{7}e^{-7s}\cos(s) + \frac{1}{49}e^{-7s}\sin(s)$$

$$-\frac{1}{49}\int e^{-7s}\cos(s)ds$$

$$\Rightarrow \frac{50}{49}\int e^{-7}\cos(s)\,ds = -\frac{1}{7}e^{-7s}\cos(s) + \frac{1}{49}e^{-7s}\sin(s)$$

$$\Rightarrow \int e^{-7}\cos(s)ds = -\frac{7}{50}e^{-7s}\cos(s) + \frac{1}{50}e^{-7s}\sin(s)$$

$$+ C$$

Thus

$$\int_0^t e^{-7s}\cos(s)\,ds = -\frac{7}{50}e^{-7t}\cos(t) + \frac{1}{50}e^{-7t}\sin(t) + \frac{7}{50}$$

In conclusion,

$$x(t) = e^{7t}x(0) + e^{7t}\int_0^t e^{-7s}\cos(s)\,ds$$

$$= \boxed{e^{7t}x(0) - \frac{7}{50}\cos(t) + \frac{1}{50}\sin(t) + \frac{7}{50}e^{7t}}$$