MATH 3323: Differential Equations

Instructor: Nestor Guillen

What we have loved, others will love,
and we will teach them how.
(William Wordsworth)

Before anything else, a little story

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Here is Newton's Universal Law of Gravitation

$$F = \frac{Gm_1m_2}{r^2}$$

(point masses attract one another with a force proportional to the product of their respective two masses and the inverse square of the distance between them)

Before anything else, a little story

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Now, seriously, how did Newton even come up with this?!

Newton's Universal Law of Gravitation

$$F = \frac{Gm_1 m_2}{r^2}$$

The discovery of this law represents a crucial point in the history of humanity. For instance,

- It predicted with unprecedented accuracy the location of planets in the sky, as well as the passing of comets.
- It led to a new era of astronomy and science in general.
- The predictions made by the theory were accurate enough to guide the navigation of the **Apollo missions**, hundreds of years after its discovery. The equation is widely used **everyday**.

Now, seriously, how did Newton even come up with this?!

$$F = \frac{Gm_1m_2}{r^2}$$

The apocryphal tale of Newton's discovery of the universal law of gravitation involved the falling of an apple.

What actually happened was a hard earned discovery which combined experimental observations by astronomers like Kepler and Brahe, and mathematics, including analytic and Greek geometry, and the newly developed differential calculus.

What was known at the time of Newton

- The Euclidean geometry of the Ancient Greeks, and importantly for us, the knowledge of conic sections.
- Descartes introduction of coordinates into geometry (a relatively recent event in Newton's days).
- Kepler's description of three empirical laws describing the behavior of the planets –based on astronomical data.

What was known at the time of Newton

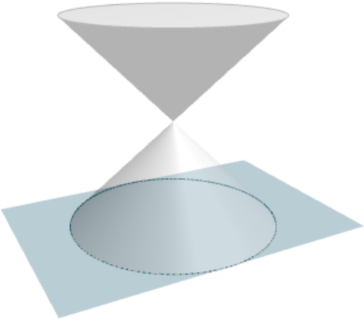
Apollonius' Conic Sections

Apollonius of Perga (presently Bergama, Turkey) was a Greek geometer and astronomer who lived in the late 3rd/early 2nd century BC. Perhaps he is best known in our time for his work on ***conic sections****.*

What was known at the time of Newton

Apollonius' Conic Sections

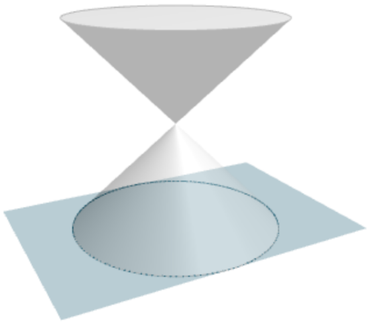
First: consider a double **cone**, just as the one below...



What was known at the time of Newton

Apollonius' Conic Sections

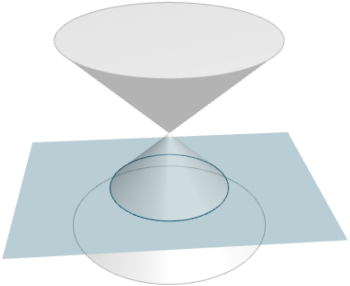
. . . as well as a **plane** that we are free to move around.



What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

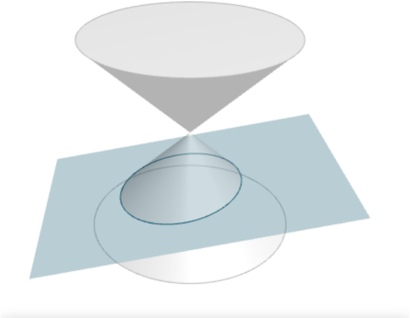


In this case, the curve turns out to be a **circle**.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

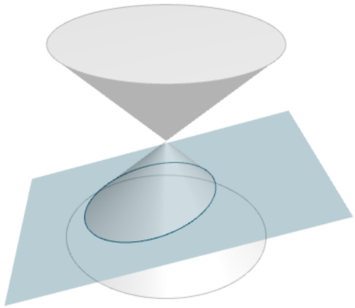


If we start tilting the plane, the circle turns into an **ellipse**.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.



The more we tilt the plane, the more **eccentric** the ellipse.

What was known at the time of Newton

Apollonius' Conic Sections

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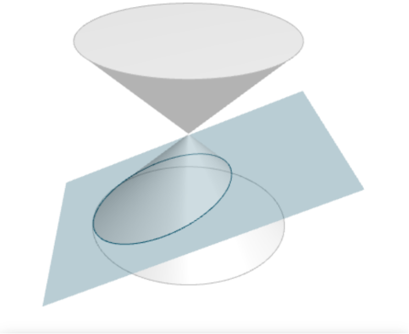

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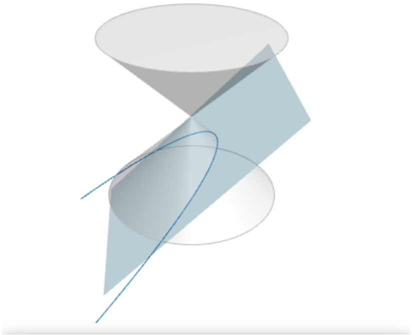
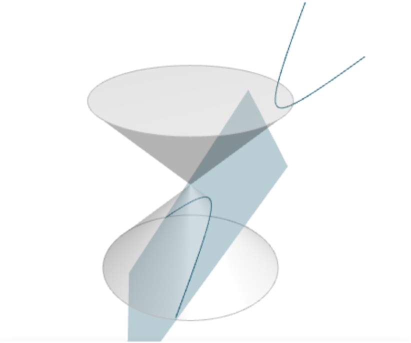


When the tilting is just right, the ellipse turns into a **parabola**!.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.

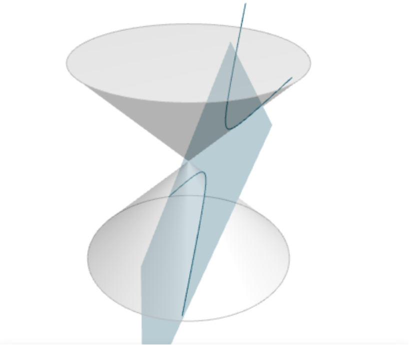


The slightest tilt turns the parabola into a **hyperbola**.

What was known at the time of Newton

Apollonius' Conic Sections

The way the plane "cuts" the cone forms a planar curve.



Further tilting gives us a more eccentric hyperbola.

What was known at the time of Newton

Apollonius' Conic Sections

So, a conic section is the planar curve formed by the intersection of a plane with a double cone. The possible shapes are ellipses, parabolas, hyperbolas, and a pair of straight lines crossing each other.

What was known at the time of Newton

Apollonius' Conic Sections

Scientists (or "natural philosophers" as they called themselves) were guided by an ideal that the laws governing the universe must be expressible in terms of mathematics, and in terms of simple rules and beautiful geometric constructions.

This was true in particular, of scientists like the German astronomer Johannes Kepler, who was part of the scientific revolution of the 17th century.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

First Law: The motion of a planet describes an ellipse, with the sun located at one of its two focal points.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

Second Law: The trajectory along the orbit is such that, if one draws a line joining the planet to the sun, and keeps track of the region swept by it over time, then the area of this region is the same for all time intervals of equal length.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

Third Law: The orbit time T for the planet is such that its square is proportional to the cube of the major semi-axis of the ellipse.

What was known at the time of Newton

Hooke's harmonic motion

Hooke attempted to explain planetary movements with

$$\ddot{x} = -k^2 x$$

this is what we now a call second order differential equation.

We will learn its solutions have a simple formula

$$x(t) = x(0)\cos(kt) + \frac{\dot{x}(0)}{k}\sin(kt)$$

What was known at the time of Newton

Description of an ellipse

The formula for the ellipse in polar coordinates is

$$r(\theta) = \frac{a(1-e^2)}{1 - e\cos(\theta)}$$

where θ is the angle given by the planet's position and

$$\begin{aligned} a &= \text{ semi-major axis} \\ b &= \text{ semi-minor axis} \\ e &= \text{ eccentricity} = \sqrt{1-(b/a)^2} \end{aligned}$$

What was known at the time of Newton

Description of an ellipse

How is θ changing with time? Here we use the **Second Law**

$$\text{Area}(\theta_1, \theta_2) = \frac{1}{2} \int_{\theta_1}^{\theta_2} r(\theta)^2 \, d\theta$$

Then, by the **Second Law**

$$\text{Area}(\theta(t), \theta(t+s)) \text{ is independent of } t.$$

Dividing by s and letting $s \to 0$, one concludes that

$$r^2(\theta(t))\dot{\theta}(t) \text{ is independent of } t$$

What was known at the time of Newton

Description of an ellipse

We now put everything in Cartesian coordinates

$$x(t) = r(\theta(t))\cos(\theta(t)), \;\; y(t) = r(\theta(t)\sin(\theta(t))$$

From the chain rule, and the relation $\dot{\theta}(t)r(\theta(t))^2 = c_0$, one gets

$$\dot{x}(t) = -c_0 \sin(\theta(t)), \;\; \dot{y}(t) = c_0(\cos(\theta(t)) - e)$$

What was known at the time of Newton

Description of an ellipse

Taking derivatives again, and using the relation between $\dot{\theta}$ and $r(\theta)$ again, one obtains

$$\ddot{x} = -c\frac{1}{r(\theta)^2}\cos(\theta),$$
$$\ddot{y} = -c\frac{1}{r(\theta)^2}\sin(\theta).$$

for some constant c. Using the **Third Law**, one sees further c is independent of e and a.

That's it. This last equation tells you the acceleration of the planet is given by the inverse square distance to the sun!

What was known at the time of Newton

One more thing: circular orbits

First Law (for a circle)

$$x(t) = R\cos(\theta(t)), \;\; y(t) = R\sin(\theta(t)).$$

Second Law

$$\theta(t) = \omega t + \theta_0.$$

Third Law

$$\omega = cR^{-3/2}$$

The inverse square law follows. This was known before Newton!

What's the story?

1. A **problem** arising from the **physical** world: what, if any, are the mechanisms determining planetary motion?.
2. Empirical **observations**: Kepler's three laws.
3. A new **mathematical tool** is developed (Calculus), allowing Newton to find the Universal Law of Gravitation, which gives a full explanation for Kepler's laws.
4. The new theory is fully captured in a **differential equation**. Solutions agree with available observations, and then are used to make **predictions**. Such predictions were considered astonishing at the time, such as Halley's prediction of the arrival of a comet.

Welcome to

MATH 3323: Differential Equations

What this class is about

Goal: To gain practice in the craft and science of using differential equations to describe, understand, and predict **things**.

Disclaimer: This is a challenging topic, but it is also an extremely rewarding and far reaching undertaking.

The study of differential equations, and their multidimensional counterparts, partial differential equations, is a life long subject, with different disciplines mastering different equations according to the phenomenon they study.

Class setup and evaluation

My office: MCS 468.
Email: nestor@txstate.edu.

Office hours (tentative): Monday-Wednesday 10:00 am-12:00 pm.

"Email Office hours": Fridays 9:00-11:00 am.

Supplemental Instructor: Kennedy Farrell.
Email: kaf124@txstate.edu

Class set up and evaluation

Evaluation:

- Problem Sets 25% (almost weekly, 14 total)
- SI sessions: 5%
- 3 Exams: 30%, 15%, 0%
(weighted by best grade, second best grade, and third best grade)
- Final 25%.
- Bonus problems: 1 or 2 per Problem Set, **worth 1% each**.

Class set up and evaluation

Problem Set Policies:

Lowest 3 grades are dropped when computing their average.

No late Problem Sets are accepted.

Class textbook:
Elementary Differential Equations and Boundary Value Problems, Boyce-DiPrima-Meade (11th ed.)

Homepage: for the syllabus, course schedule, additional notes, and problem sets.

https://ndguillen.github.io/math3323.html

All information here will also be available on TRACS.

Class set up and evaluation

Important Dates

Problem sets will always be due on Monday, except for Problem Set 1, which will be due on Wednesday September 4th.
Exams will be on Mondays: September 30th, October 21st, and November 18th.
Final exam time TBA.

Class set up and evaluation

Supplemental Instruction

Supplemental Instruction (SI) is a nontraditional form of tutoring provided by SLAC and Ingram School of Engineering that focuses on collaboration, group study, and interaction for assisting students in difficult courses. This program provides a trained peer who has already successfully completed the course to assist you. This peer, called the SI Leader, will attend this class each day, participate as any normal student (takes notes, exams, etc.), and then facilitate several one-hour study sessions per week for group study.

Supplemental Instruction

Because of the positive impact SI participation makes on course grade, you will be required to attend 10 SI sessions (once each week) by Monday, November 11. This will make up 5% of your course grade. You may attend no more than 2 hours of session the week of November 3-8, so please plan to attend one session each week. You will select a session to attend each week via a survey that will be emailed to you. Be sure to make note of the day/time/location of the session you select.

If you have concerns regarding the SI program or wish to verify your number of sessions attended, please contact the Program Coordinators, Lindley Alyea (lindley@txstate.edu 512-245-2515) or Victor Capellan (victor@txstate.edu 512-245-2515)..

Differential Equations:
Basic terms and examples

Differential equations and where they are used

The harmonic oscillator (pendulums, springs, circuits)

The nonlinear pendulum (more accurate model for a pendulum)

The N-body problem (multiple planet dynamics)

The exponential growth/decay equation (population growth, radioactive decay, compound interest)

The logistic equation (population growth)

Euler, Lagrange, and Kovalevskaya tops (spinning tops!)

The Lorenz attractor (meteorology)

The Lotka-Volterra equation (predator-prey models in biology)

Basic terms

In each of the examples above, one describes the state of the system at time t, via a vector made out of d-coordinates

$$x(t) = (x_1(t), \ldots, x_N(t))$$

The number of coordinates needed to describe the state of a system in a given instant is called the **dimension of the system.**

The description must be such that, roughly speaking, one has included all the relevant information about the system necessary to determine its instantaneous evolution.

In this class 99.99% of the time we will have $N = 1$ or $N = 2$.

Examples

The exponential growth/decay equation: the current amount of the quantity considered $x(t)$.

$$\dot{x} = \lambda x$$

Examples

The logistic equation: the size of the population $p(t)$.

$$\dot{p}(t) = rp(t)(1 - \frac{p(t)}{M})$$

Examples

The harmonic oscillator: if $x(t)$ is the displacement then

$$\ddot{x} = -\kappa^2 x$$

if one introduces the displacement velocity $y(t)$ this equation can be rewritten as a system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\kappa^2 x \end{aligned}$$

Some of what we discussed today

1. Differential equations are essential to the modern understanding of much of the sciences and engineering.
2. Explicit formulas that represent a solution are useful – when you can find them.
3. Systems admitting an explicit formula are the exception and not the rule.
4. The higher the dimension of the problem, and the more nonlinear it is, the harder it is to analyze.
5. The more complicated a phenomenon, the higher dimensional and the more non-linear the differential equations required to describe it.