Texas State University
MATH 3323: Differential Equations
Instructor: Nestor Guillen

Problem Set 9

(1) Find the general solution to $\dot{x} = Ax + b$ for each A and b given bellow (note you are explicitly given the exponential matrix e^{tA} for each case)

(a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$, $e^{tA} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$

(b) $A = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -2/3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ -3t \end{pmatrix}$, $e^{tA} = \begin{pmatrix} e^{(\sqrt{3})t} & 0 \\ 0 & e^{-(2/3)t} \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 5e^{3t} \\ 10e^{3t} \end{pmatrix}$, $e^{tA} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$

(2) Find the general solution to $\dot{x} = Ax + b$, where

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

Hint: Remember that the general solution is a sum of the general solution to the homogeneous equation plus a particular solution to the inhomogeneous equation.

(3) Solve the IVP for each linear second order equation given bellow

(a) $3\ddot{x} - \dot{x} + 2x = 0,\ x(0) = 2,\ \dot{x}(0) = 0$
(b) $5\ddot{x} + 2\dot{x} + 7x = 0,\ x(0) = 2,\ \dot{x}(0) = 1$
(c) $\ddot{x} + 2\dot{x} + 6\dot{x} = 0,\ x(0) = 2, \dot{x}(0) = -1$

(4) Consider the following combination of two cosines and sines with frequency k

$$x(t) = A\cos(kt) + B\sin(kt)$$

The point of this exercise is discovering that for any A and B the resulting function $x(t)$ is simply a shift of $\cos(kt)$ by some amount determined by R.

(a) Assume that A and B are such that $A^2 + B^2 = 1$, show then that for some angle δ we have

$$x(t) = \cos(kt - \delta).$$

Hint: To find δ, make use of the well known formula for the cosine of a difference

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

(b) Show that given any A and B you can find numbers R and δ such that

$$x(t) = R\cos(kt - \delta).$$

Hint: Multiply both A and B by some properly chosen factor so that the sum of the squares is 1, and use part a).

(5) (BONUS) Consider a linear second order differential equation

$$\ddot{x} + B\dot{x} + Cx = 0,$$

such that $B^2 - 4C < 0$. In this case the roots of the characteristic polynomial are not real numbers.

(a) Show that if the roots are $\lambda = \mu \pm i\omega$ then μ and ω are given by the formulas

$$\mu = -\tfrac{1}{2}B \text{ and } \omega = \tfrac{1}{2}\sqrt{4C - B^4}$$

(b) Write down a general formula for solutions $x(t)$ using μ and ω.

(c) Let $x(t)$ be a real solution of the differential equation, using the formula obtained in part (b) show that the function $x(t)$ will always take both positive and negative values as t varies, except if x is the trivial solution given by $x(t) = 0$ for all t.

(6) (BONUS) Go to the link

`https://mybinder.org/v2/gh/ndguillen/3323_Sp2020/master`

and open the notebook titled "2D_LinearSystems_MatrixExponential". Run the code as explained in class in order to generate solutions to linear systems following the instructions bellow.

(a) Plot the trajectory starting from $x_0 = (5, 0)$ for the matrix

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$$

For various values of α: 0.1,0.5,1,2. In a couple of sentences, answer the following: How does the trajectory change as you go over increasing values of α? Do the same but for the values $-0.1, -0.5, -1, 2$, how did this change the trajectories?

(b) Now consider $x_0 = (1, 0)$ the matrix

$$A = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix}$$

For various values of β: $0, 0.1, 0.5, 1, 2$. In a couple of sentences, answer the following: How does the trajectory change as you increase β?

(c) Do the same as in the previous step, but changing the initial data to $x_0 = (10, 0)$ and taking β equal to $-0.1, -0.5, -1, -2$. How did the negative sign change the behavior?

(d) Plot the trajectory starting from $x_0 = (5, 0)$ for the matrix

$$A = \begin{pmatrix} 0 & -\alpha \\ 1 & 0 \end{pmatrix}$$

where α takes the values $0.1, 0.5$, and 2. How does the size of α with respect to 1 affect the shape of the trajectory? What will happen to the trajectory if for one of these matrices we change the diagonal elements from 0 to a small negative number, say -0.5?

(7) (BONUS) In the context of the previous problem, give an example of a 2x2 matrix such that the trajectories describe an ellipse (and not a circle) where the largest axis is parallel to the vector $(1, 1)$. *Hint:* Consider the last part of the previous exercise, and then use a linear transformation to rotate things by 45 degrees.