Texas State University
MATH 3323: Differential Equations
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Problem Set 8

This Problem Set is due on April 14th, remember to read the guidelines to submit problem sets via email. This covers material discussed in the 4 lectures after Spring Break. Relevant book sections: 7.5, 7.6, and 7.7.

(1) Compute the following matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

After considering the outcome of this product, conclude that if $ad - bc \neq 0$ then the matrix on the left has an inverse matrix. Provide a formula for this inverse using a, b, c, d.

(2) For each matrix A bellow, compute e^{tA}. In each case you are given a basis of eigenvectors for A that you can use in your computation (the respective eigenvalues are not given, but those you can find easily given the eigenvectors!).

(a) $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & -1 \\ 10 & -5 \end{pmatrix}, \ v_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $A = \begin{pmatrix} -2 & 8 \\ 2 & -2 \end{pmatrix}, \ v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(3) In each case, take the given complex-valued solutions to the differential equation and write separately their real and imaginary parts, that is, write each given $x(t)$ as

$$x(t) = x_1(t) + ix_2(t)$$

where the vector functions $x_1(t)$ and $x_2(t)$ only take real values. Then, in each case check that the real and imaginary parts solve the linear system (by plugging them into the equation), and use them to write the general solution to the system.

(a) $\dot{x} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} x, \quad x(t) = \begin{pmatrix} ie^{4it} \\ e^{4it} \end{pmatrix},$

(b) $\dot{x} = \begin{pmatrix} 2 & -6 \\ 6 & 2 \end{pmatrix} x, \quad x(t) = \begin{pmatrix} -e^{(2+6i)t} \\ ie^{(2+6i)t} \end{pmatrix},$

(c) $\dot{x} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} x, \quad x(t) = \begin{pmatrix} ie^{(2-3i)t} \\ e^{(2-3i)t} \end{pmatrix}$

(4) Find the general (real valued) solution to the following two systems

$$\text{(a) } \dot{x} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 4 \end{pmatrix} x \qquad \text{(b) } \dot{x} = \begin{pmatrix} -4 & 1 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix} x$$

Then, find the respective fundamental matrices $\Psi(t)$ with $\Psi(0) =$ the identity matrix.
Note: Don't forget to write the final answer with solutions taking only real values.

(5) (BONUS) Find a fundamental matrix $\Psi(t)$ (for example, e^{tA}) for the system $\dot{x} = Ax$ for each of the following A's.

$$\text{(a) } A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \qquad \text{(b) } A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

(6) (BONUS) Consider the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Let P denote the matrix $P = A - 3\mathrm{I}$ (I denotes the identity matrix). Then

(a) Show by direct computation that P^2 is the zero matrix – i.e. all its entries are zero.
(b) Compute a formula for e^{tP} (since $(tP)^2 = 0$, the series simplifies considerably!)
(c) Use this to compute a simple formula for e^{tA}.