Texas State University
MATH 3323: Differential Equations
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Problem Set 7

This Problem Set is due on April 7th, remember to read the guidelines to submit problem sets via email. This covers material discussed in class before Spring break, in particular the lectures on March 10th and 12th. Relevant book sections: 7.2, 7.3, 7.4, and 7.5.

(1) In each item below, compute the derivative of the given vector-valued function $\mathbf{x}$ (whose components are denoted $x_1(t)$ and $x_2(t)$) and match it to the differential equation it solves from the list on the right column

i) $\mathbf{x}(t) = \begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$ a) $\dot{x} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} x$

ii) $\mathbf{x}(t) = \begin{pmatrix} e^{2t}(t+1) \\ e^{2t} \end{pmatrix}$ b) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) + \frac{\sqrt{2}}{2} \end{cases}$

iii) $\mathbf{x}(t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$ c) $\dot{x} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} x + \begin{pmatrix} -2t \\ 1 \end{pmatrix}$

iv) $\mathbf{x}(t) = \begin{pmatrix} t + \frac{1}{2}(e^{2t} - 1) \\ \frac{1}{3}(e^{3t} - 1) \end{pmatrix}$ d) $\dot{x} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$

(2) For each part of this problem you are given a two dimensional differential equation and two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of said equation, in each case check that the given solutions are linearly independent at $t = 0$, and use them to find a solution $\mathbf{x}$ to the differential equation with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

a) $\dot{\mathbf{x}} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}_1(t) = \begin{pmatrix} \cos(4t) \\ \sin(4t) \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} \sin(4t) \\ -\cos(4t) \end{pmatrix}$.

b) $\dot{\mathbf{x}} = \begin{pmatrix} 10 & 0 \\ 0 & -10 \end{pmatrix} \mathbf{x}$, $\mathbf{x}_1(t) = \begin{pmatrix} e^{10t} \\ 0 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} 0 \\ e^{-10t} \end{pmatrix}$.

c) $\dot{\mathbf{x}} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x}$, $\mathbf{x}_1(t) = \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} 3e^{3t} \\ 0 \end{pmatrix}$.

(3) Find the solution to $\dot{x} = Ax$ with initial value $x(0) = x_0$ for each A and x_0 given below

(a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$, $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (b) $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$, $x_0 = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$,

(c) $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 7 & 3 \end{pmatrix}$, $x_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, (d) $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 3 & 1 \end{pmatrix}$, $x_0 = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$,

Compare the answers for (a),(b) with those for (c), (d) respectively.

(4) For each pair of matrices A and $\Psi(t)$ given bellow compute $\dot{\Psi}(t)$ and $A\Psi(t)$.

(a) $A = \begin{pmatrix} -10 & 0 \\ 0 & 5 \end{pmatrix}$, $\Psi(t) = \begin{pmatrix} e^{-10t} & 0 \\ 0 & e^{5t} \end{pmatrix}$

(b) $A = \begin{pmatrix} -10 & 0 \\ 0 & 5 \end{pmatrix}$, $\Psi(t) = \begin{pmatrix} e^{-10t} & -e^{-10t} \\ 3e^{5t} & -2e^{5t} \end{pmatrix}$

(c) $A = \begin{pmatrix} 0 & -8 & 0 \\ 8 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $\Psi(t) = \begin{pmatrix} \cos(8t) & -\sin(8t) & 0 \\ \sin(8t) & \cos(8t) & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$

(5) (BONUS) For each linear second order system below, write it as first order system and use that representation to find the general solution

(a) $\ddot{x} - 4x = 0$,

(c) $4\ddot{x} - 4\dot{x} - 3x = 0$,

(d) $4\ddot{x} + 17\dot{x} + 4x = 0$.

(6) (BONUS) Find the general solution to the system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & -4 & 7 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$