Texas State University
MATH 3323: Differential Equations
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Problem Set 5

(1) For each family of vectors, determine wether the vectors are linearly independent or not, and in case they are linearly dependent, find a linear relation between them.

a) $\mathbf{x}_1 = (2, 2, 0), \ \ \mathbf{x}_2 = (0, -2, 2), \ \ \mathbf{x}_3 = (1, 0, 1)$
b) $\mathbf{x}_1 = (1, 1, 0), \ \ \mathbf{x}_2 = (0, 1, 1), \ \ \mathbf{x}_3 = (1, 0, 1)$
c) $\mathbf{x}_1 = (1, 1, 0), \ \ \mathbf{x}_2 = (1, 2, 1), \ \ \mathbf{x}_3 = (0, 1, 1)$

(2) Consider the real valued functions $x_1(t) = e^{-2t}$ and $x_2(t) = e^{-3t}$, then
(a) By direct computation, check that each function solves

$$\ddot{x} + 5\dot{x} + 6x = 0$$

(b) Find α_1 and α_2 such that the function given by

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

solves

$$\begin{aligned} &\ddot{x} + 5\dot{x} + 6x = 0 \\ &x(0) = 2, \ \dot{x}(0) = 9. \end{aligned}$$

(3) Check whether the following families of functions of t are linearly independent or not

(a) $t^2 + 1, \ 2t, \ 4(t+1)^2$
(b) $\sin(t)\cos(t), \ \sin(2t) + \cos(2t), \ \cos(2t)$
(c) $t^2 + 2t, 10t, -2t^2 + 3t$
(d) $\dfrac{1}{t^2 - 1}, \ \dfrac{1}{t+1}, \ \dfrac{1}{t-1}$

(4) Consider the two vector valued functions

$$\mathbf{x}_1(t) = (2e^t, 3) \text{ and } \mathbf{x}_1(t) = (4, 6e^{-t}).$$

For any given fixed value t_0, show that the two dimensional vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly dependent. At the same time, show that $\mathbf{x}_1$ and $\mathbf{x}_2$ as functions of t are linearly independent.

(5) (BONUS) Consider the function

$$x(t) = e^{2t}$$

Using the chain rule, show that given any numbers A, B, and C, then for all t we have the formula

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = (A4 + B2 + C)x$$

Determine similar formulas for $A\ddot{x}(t) + B\dot{x}(t) + Cx(t)$ for the following functions

$$e^t, \ e^{-t}, \ e^{3t}, \ e^{-3t}.$$

Based on your findings, try to find a (non-zero) function $x(t)$ solving the equation

$$\ddot{x} - 5\dot{x} - 24x = 0.$$

(6) (BONUS) Consider a two dimensional nonlinear system given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -H'(x) \end{aligned}$$

If $(x(t), y(t))$ is any solution to this, show that there is some constant c such that

$$\frac{1}{2}y(t)^2 + H(x(t)) = c \text{ for all } t.$$

Apply this to the following problem: you are given a physical pendulum whose dynamics (θ = angle, ω = angular velocity) are governed by the equations

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -A\sin(\theta). \end{aligned}$$

Suppose the pendulum starts from a horizontal position (i.e. lying at an angle of $\pi/2$ away from equilibrium) and is initially at rest (i.e. the initial angular velocity is zero). What is the angular velocity at the moment when the pendulum is perfectly vertical?