Texas State University
MATH 3323: Differential Equations
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Problem Set 3

This problem set deals with Sections 2.1 and 2.2, the new method being considered is separation of variables. The fourth problem is a review of solutiosn to linear systems of equations (using for example the standard elimination method), but if you feel like it you may also check out the equivalent method described at the beginning of Section 7.3.

(1) Use separation of variables to solve problems 1 through 5 at the end of Section 2.2 –you may use the answers at the end of the book to check your solution, but as always, you must show your work to get credit for the question.

(2) Find the solution $x(t)$ to each differential equation taking the indicated value

$$\text{a) } \dot{x} = (1-2t)x^2, \quad x(0) = -3$$

$$\text{b) } \dot{x} = \frac{tx}{\sqrt{1+t^2}}, \quad x(0) = 1$$

$$\text{c) } \dot{x} = 5t^{-1}x, \quad x(0) = 5$$

(3) Find the function y with $y(0) = 0$ which solves the equation

$$y'(x) = \frac{2-e^x}{3+2y(x)}$$

Once you find $y(x)$, find the value of x where y attains its maximum value.

(4) Solve each system of equations below (i.e. if there are solutions find all of them, or else explain why there aren't any solutions)

$$\text{a) } \begin{cases} 9x_1 & -x_2 & = 3 \\ 3x_1 & +x_2 & = 1 \end{cases}$$

$$\text{b) } \begin{cases} x_1 & +x_2 & -x_3 & = 1 \\ 2x_1 & +x_2 & +x_3 & = 1 \\ x_1 & -x_2 & +2x_3 & = 1 \end{cases}$$

$$\text{c) } \begin{cases} x_1 & & -x_3 & = 0 \\ 3x_1 & +x_2 & +x_3 & = 1 \\ -x_1 & +x_2 & +2x_3 & = 2 \end{cases}$$

(5) (BONUS) Let $x(t)$ be a positive function satisfying the **inequality**

$$\dot{x}(t) \leq -\lambda x(t) \text{ for all } t$$

for some number $\lambda > 0$. Show that

$$x(t) \leq x(0)e^{-\lambda t} \text{ for } t > 0.$$

Discuss: what is the relationship between the value of λ and the behavior of $x(t)$ as t goes to infinity? (for example, take $x_1(t)$ and $x_2(t)$ functions as above with two different constants λ_1 and λ_2, what can you say about $x_1(t)/x_2(t)$ as $t \to \infty$ if $\lambda_1 > \lambda_2$?).

(6) (BONUS) Let $x_1(t)$ and $x_2(t)$ be two solutions to the equation

$$\dot{x} = f(x),$$

where all know about $f(x)$ is there is some $L > 0$ such that whenever $a > b$ we have

$$f(a) - f(b) \leq -L(a - b)$$

Show the following

(a) For any pair a, b (regardless of which is larger) we have the inequality

$$(a - b)(f(a) - f(b)) \leq -L(a - b)^2$$

(b) Given $x_1(t)$ and $x_2(t)$ we have the "differential inequality"

$$\frac{d}{dt}(x_1 - x_2)^2 \leq -2L(x_1 - x_2)^2.$$

Use this to show the following inequality for solutions of the original differential equation

$$|x_1(t) - x_2(t)| \leq e^{-Lt}|x_1(0) - x_2(0)|, \quad \text{for } t > 0.$$

What do you think is the significance of this inequality? For the sake of concreteness, think for a second $x_1(t)$ and $x_2(t)$ represent the state of some physical system, what does this last inequality say about the behavior of the state of the system as time increases?

Hint: Once you obtain the differential inequality, use the previous bonus problem.