March 9th, 2020

Eigenvalues and eigenvectors

As we said at the end of the previous lecture, given a square matrix A a vector x is called an eigenvector of A if x is not the zero vector and

$$Ax = \lambda x$$

for some number $\lambda \in \mathbb{R}$. In such a case we say λ is an eigenvalue of A, and the set of all eigenvalues of A is called the spectrum of A, and is denoted $\sigma(A)$.

So an eigenvector of A is one vector x which when multiplied by A results in a parallel vector, with λ denoting the factor by which we are "stretching" or "shrinking" the vector x.

Example: Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Then $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+1 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thus for this matrix A the vector $\begin{pmatrix}1\\1\end{pmatrix}$ is $\big($
an eigenvector with eigenvalue $\lambda = 3$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ is
an eigenvector with eigenvalue $\lambda = 1$.

Example (slightly more complicated situation)

Consider
$$A = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

Observe
$$A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

so $\begin{pmatrix}1\\2\\1\end{pmatrix}$ is an eigenvector of A with eigenvalue λ.

Eigenvalues and eigenvectors tell us a lot about a matrix/linear transformation (in fact, sometimes they tell you all there is to know about the matrix)

Moreover, linear system of differential equations can be solved if one can find all the eigenvalues an eigenvectors of matrices.

~~[illegible]~~

To illustrate this, let us talk again about linear systems in the context of matrices and vectors, and run an example.

Note that in terms of matrix and column vector notation, a linear system of differential equations can be written simply as

$$\dot{x} = Ax$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ and the ~~coefficients~~ entries of A correspond to the coefficients appearing in the equations of the system.

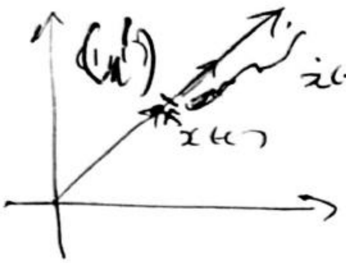
<u>Example</u>: Let us consider the system

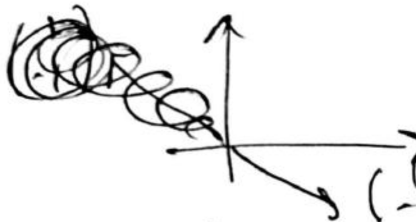
$$\dot{x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x$$

(This by the way corresponds to

$$\dot{x}_1 = 2x_1 + x_2$$
$$\dot{x}_2 = x_1 + 2x_2$$)

We saw earlier that $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors corresponding to $\lambda = 3$ and $\lambda = 1$, respectively.





If a solution to the system lies at $\binom{1}{1}$ at some instant, then it will move along the same direction, according to

$$\dot{x} = 3x$$

(at least for an instant)

An instant later, we still expect the solution to lie along the straight line spanned by $\binom{1}{1}$. So it makes sense to <u>guess</u> that a solution to the system takes the form

$$x(t) = h(t)\begin{pmatrix}1\\1\end{pmatrix} \quad (*)$$

for a scalar function $h(t)$... but what function would make $(*)$ a solution? well, let's substitute this into the equation and see what condition we get on h!

$$\dot{x} = \begin{pmatrix}2&1\\1&2\end{pmatrix}x$$

becomes

$$\begin{aligned}\dot{h}(t)\begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}2&1\\1&2\end{pmatrix}h(t)\begin{pmatrix}1\\1\end{pmatrix}\\ &= h(t)\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ &= 3h(t)\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}$$

So, for $h(t)\binom{1}{1}$ to be a solution, all we need is that $\boxed{\dot{h} = 3h}$, so $h(t) = e^{3t}$ works!

What does this example show us? Let's summarize: if v is an eigenvector of A with eigenvalue λ, then the function (vector)

$$x(t) = e^{\lambda t} v$$

is a ~~[illegible]~~ solution of

$$\dot{x} = Ax.$$

Moreover, if we know that $v_1, v_2, \ldots, v_k$ are eigenvectors of A with respective eigenvalues $\lambda_1, \ldots, \lambda_k$, then each of the functions (vector)

$$x_1(t) = e^{\lambda_1 t} v_1$$

$$\vdots$$

$$x_k(t) = e^{\lambda_k t} v_k$$

solves $\dot{x} = Ax$, and by linear superposition, for every constant of numbers $c_1, \ldots, c_k$, the function

$$c_1 e^{\lambda_1 t} v_1 + \cdots c_k e^{\lambda_k t} v_k$$

will be a solution to the differential equation.

Thus, we have learned: <u>eigenvalues and eigenvectors can be very useful when it comes to solving differential equations!</u>

<u>Example</u>: Revisiting

$$\dot{x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x$$

we saw $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors with eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$ respectively. We then see that any function of the form

$$x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(alternatively written as

$$x(t) = \begin{pmatrix} c_1 e^{3t} + c_2 e^{t} \\ c_1 e^{3t} - c_2 e^{t} \end{pmatrix} \Big)$$

is a solution to $\dot{x} = Ax$. What's more! all solutions can be obtained in this way.

<u>Finding eigenvalues and eigenvectors</u>.

Suppose you know for a fact that a number λ is an eigenvalue of A.

This means it is possible to solve the equation

$$Ax = \lambda x$$

via a non-zero vector x.

So

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ & & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

This is the same as asking that

$$\begin{pmatrix} A_{11}-\lambda & A_{12} & \cdots & & A_{1n} \\ A_{21} & A_{22}-\lambda & & & \\ & & A_{33}-\lambda & & \\ \vdots & & & \ddots & \\ A_{n1} & & & & A_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

or, $\quad (A-\lambda I)x = 0 \quad$ has non-trivial solutions.

(here, $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$).

<u>Example</u>: Let's revisit $A = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}$,

where we know $\lambda = 6$ is an eigenvalue.

$$A - 6I = \begin{pmatrix} 2-6 & 1 & 2 \\ 4 & 2-6 & 4 \\ 2 & 1 & 2-6 \end{pmatrix} = \begin{pmatrix} -4 & 1 & 2 \\ 4 & -4 & 4 \\ 2 & 1 & -4 \end{pmatrix}$$

Solving the system $(A-6I)x = 0$ we see that if

$$x_3 = x_1, \quad x_2 = 2x_1$$

then (x_1, x_2, x_3) will be a solution of the system regardless of the value of x_1

choosing $x_1 = 1$ we get the eigenvect $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ mentioned previously.

So, once we know λ is an eigenvalue, finding eigenvectors for λ amounts to solving a basic linear algebraic system of equations — or equivalently, finding the kernel of $A - \lambda I$.

The next obvious question is then, how do we determine the eigenvalues of A?