MATH 3323: Differential Equations

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Nothing in life is to be feared, it is only to be understood.
Now is the time to understand more, so that we may fear less.
- Marie Curie

Differential Equations

what they are and how to solve them

First, let us talk a bit about the movement of the planets.

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Here is Newton's Universal Law of Gravitation

$$F = \frac{Gm_1m_2}{r^2}$$

(two point masses have accelerate towards each other proportionally to the inverse square of the distance between them)

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- It predicted with unprecedented accuracy the location of planets in the sky, as well as the passing of comets.
- It marks the beginning of modern physics based on the power of mathematics to make useful **predictions**
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Most relevant to us, this law is a **differential equation**.

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The apocryphal tale of Newton's discovery of the universal law of gravitation involved the falling of an apple.

What actually happened was a hard earned discovery which combined experimental observations by astronomers like Kepler and Brahe, and mathematics, including analytic and Greek geometry, and the newly developed differential calculus.

What was known at the time of Newton

- The Euclidean geometry of the Ancient Greeks, and importantly for us, the knowledge of conic sections.
- Descartes introduction of coordinates into geometry (a relatively recent event in Newton's days).
- Kepler's description of three empirical laws describing the behavior of the planets –based on astronomical data.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

First Law: The motion of a planet describes an ellipse, with the sun located at one of its two focal points.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

Second Law: The trajectory along the orbit is such that, if one draws a line joining the planet to the sun, and keeps track of the region swept by it over time, then the area of this region is the same for all time intervals of equal length.

What was known at the time of Newton

Kepler's Laws of Planetary Motion

Third Law: The orbit time T for the planet is such that its square is proportional to the cube of the major semi-axis of the ellipse.

What was known at the time of Newton

Hooke's harmonic motion

Hooke attempted to explain planetary movements with

$$\ddot{x} = -k^2 x$$

this is what we now a call second order differential equation.

We will learn its solutions have a simple formula

$$x(t) = x(0)\cos(kt) + \frac{\dot{x}(0)}{k}\sin(kt)$$

What was known at the time of Newton

Description of an ellipse

The formula for the ellipse in polar coordinates is

$$r(\theta) = \frac{a(1 - e^2)}{1 - e\cos(\theta)}$$

where θ is the angle given by the planet's position and

$$\begin{aligned} a &= \text{ semi-major axis} \\ b &= \text{ semi-minor axis} \\ e &= \text{ eccentricity} = \sqrt{1 - (b/a)^2} \end{aligned}$$

What was known at the time of Newton

Description of an ellipse

Using the polar representation of an ellipse and calculus, Newton was able to deduce from Kepler's three laws that the planar coordinates $x(t)$ and $y(t)$ of an orbiting planet satisfy the equation

$$\ddot{x} = -c\frac{1}{r(\theta)^2}\cos(\theta),$$
$$\ddot{y} = -c\frac{1}{r(\theta)^2}\sin(\theta).$$

for some constant c independent of the ellipse!.

That's it. This last equation tells you the acceleration of the planet is given by the inverse square distance to the sun!

What was known at the time of Newton

Circular orbits are contained in the theory

First Law (for a circle)

$$x(t) = R\cos(\theta(t)), \;\; y(t) = R\sin(\theta(t)).$$

Second Law

$$\theta(t) = \omega t + \theta_0.$$

Third Law

$$\omega = cR^{-3/2}$$

The inverse square law follows. This was known before Newton!

What's the story?

1. A **problem** arising from the **physical** world: what, if any, are the mechanisms determining planetary motion?.
2. Empirical **observations**: Kepler's three laws.
3. A new **mathematical tool** is developed (Calculus), allowing Newton to find the Universal Law of Gravitation, which gives a full explanation for Kepler's laws.
4. The new theory is fully captured in a **differential equation**. Solutions agree with available observations, and then are used to make **predictions**. Such predictions were considered astonishing at the time, such as Halley's prediction of the arrival of a comet.

Welcome to
MATH 3323: Differential Equations

What this class is about

Goal: To gain practice in the craft and science of using differential equations to describe, understand, and predict **things**.

Disclaimer: This is a challenging topic, but it is also an extremely rewarding and far reaching undertaking.

The study of differential equations, and their multidimensional counterparts, partial differential equations, is a life long subject, with different disciplines mastering different equations according to the phenomenon they study.

Class setup and evaluation

My office: MCS 468.
Email: nestor@txstate.edu.

Office hours (tentative): Tuesday-Thursday 2:00 pm-3:20 pm.

"Email Office hours": Fridays 9:00-11:00 am.

Class set up and evaluation

Evaluation:

- Problem Sets 30% (almost weekly, about 14 total)

- 3 Exams: 30%, 15%, 0%
(weighted by best grade, second best grade, and third best grade)

- Final 25%.

- Bonus problems: 2 per Problem Set, **worth 1% each**.

Class set up and evaluation

Problem Set Policies:

Lowest 4 (FOUR) grades are dropped when computing their average.

No late Problem Sets are accepted.

Class textbook:
Elementary Differential Equations and Boundary Value Problems, Boyce-DiPrima-Meade (11th ed.)

Homepage: for the syllabus, course schedule, additional notes, and problem sets.

https://ndguillen.github.io/math3323.html

Webpage is also accessible from CANVAS.

Class set up and evaluation

Important Dates

Problem sets will always be typically due on Thursdays,.

Exams will be on Thursdays: February 27th, March 26th, and April 23rd.

Roster certification: February 12th.

Final exam time TBA.

Differential Equations:
Basic terms and examples

Differential equations and where they are used

The harmonic oscillator (pendulums, springs, circuits)

The nonlinear pendulum (more accurate model for a pendulum)

The N-body problem (multiple planet dynamics)

The exponential growth/decay equation (population growth, radioactive decay, compound interest)

The logistic equation (population growth)

Euler, Lagrange, and Kovalevskaya tops (spinning tops!)

The Lorenz attractor (meteorology)

The Lotka-Volterra equation (predator-prey models in biology)

Notation – functions and variables

Since the letters used for functions and independent variables varies with context, you will often see x used as an independent variable

$$f(x), y(x), g(x)$$

and other times t will denote the indepedent variable (most commonly denoting time)

$$f(t), y(t), g(t)$$

Notation – derivatives

Simiarly, the way we denote derivatives depends on context, with the following symbols

$$f'(x)\ \frac{df(x)}{dx},\ D_x f(x),\ \dot{f}(x)$$

used to represent the first derivative of a function f (so they all mean the same thing). The notation $\dot{f}$ is most commonly associated with time, and one writes

$$f'(t), \frac{df(t)}{dt},\ D_t f(t),\ \dot{f}(t)$$

Notation – derivatives

Second derivatives and higher are denoted in the following manner

$$f''(x)\ \frac{d^2 f(x)}{dx^2},\ D_t^2 f(t),\ \ddot{f}(t)$$

and for higher derivatives

$$f'''(x)\ \frac{d^3 f(x)}{dx^3},\ D_t^3 f(t),\ \dddot{f}(t)$$

and so on…

Basic terms

Typically, one describes the state of the system at time t via a "vector" made out of d-coordinates

$$x(t) = (x_1(t), \ldots, x_d(t))$$

The number of coordinates needed to describe the state of a system in a given instant is called the **dimension of the system.**

The description must be such that, roughly speaking, one has included all the relevant information about the system necessary to determine its instantaneous evolution.

In this class 99.99% of the time we will have $N = 1$ or $N = 2$.

Examples

The exponential growth/decay equation: the current amount of the quantity considered $x(t)$.

$$\dot{x} = \lambda x$$

Examples

The logistic equation: the size of the population $p(t)$.

$$\dot{p}(t) = rp(t)(1 - \frac{p(t)}{M})$$

Examples

The harmonic oscillator: if $x(t)$ is the displacement then

$$\ddot{x} = -\kappa^2 x$$

if one introduces the displacement velocity $y(t)$ this equation can be rewritten as a system

$$\dot{x} = y$$
$$\dot{y} = -\kappa^2 x$$

Some of what we discussed today

1. Differential equations are essential to the modern understanding of much of the sciences and engineering.
2. Explicit formulas that represent a solution are useful – when you can find them.
3. Systems admitting an explicit formula are the exception and not the rule.
4. The higher the dimension of the problem, and the more nonlinear it is, the harder it is to analyze.
5. The more complicated a phenomenon, the higher dimensional and the more non-linear the differential equations required to describe it.